1902-1903

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE W. E. BYERLY H. S. WHITE
F. S. WOODS MAXIME BÔCHER
E. V. HUNTINGTON

PUBLISHED UNDER THE AUSPICES OF HARVARD UNIVERSITY

SECOND SERIES Vol. 4

FOR SALE BY
THE PUBLICATION OFFICE OF HARVARD UNIVERSITY
2 University Hall, Cambridge, Mass., U. S. A.

London: LONGMANS, GREEN & Co.
39 Paternoster Row

Leipzig: OTTO HARRASSOWITZ
Querstrasse 14

LIBRARY
LELAND STANFORD JUNIOR
UNIVERSITY

THE GEODESIC LINES ON THE ANCHOR RING*

By Gilbert Ames Bliss

Mangoldt† has classified the points upon any surface according to the character of the geodesic lines passing through them. If every geodesic line through a given point remains a minimum throughout its entire length, the point is said to be of the first kind, otherwise of the second kind. He found some interesting results. For example, while upon surfaces for which the curvature is everywhere negative Jacobi has shown the points to be all of the first kind, upon those for which the curvature is everywhere positive, Mangoldt finds that points of the second kind always exist. In the latter case if there are points of the first kind the surface itself can not be closed, and all such points lie in a finite region. These regions have been investigated by Mangoldt‡ and Braunmühl§ for surfaces of the second degree which have positive curvature.

The object of the present paper is the consideration of the geodesic lines upon the anchor ring, a surface for which the curvature is positive at some points and negative at others. The geometrical character of the lines will be discussed, and the points of the ring classified according to Mangoldt's scheme.

1. The Geometrical and Analytical Definitions of Conjugate Points. Two points a and a' upon a geodesic line are said to be conjugate to each other if every arc ab, where b lies between a and a', is a minimum with respect to all the lines of its neighborhood joining a and b, and if also the arc ab ceases to be a minimum as soon as b lies beyond a'. The envelope of the geodesic lines through a touches the given line ab at the point a', and the arc aa' is not a minimum unless this envelope has a singular point of a peculiar kind at a'.‖ Weierstrass has shown¶ that when a second geodesic line

* Read before the Chicago Section of the American Mathematical Society, Dec. 27, 1900.

† Geodätische Linien auf positiv gekrümmten Flächen, *Journal für Mathematik*, vol. 91, (1881), p. 23.

‡ *l. c.*

§ Uber Envellopen geodätischer Linien, *Mathematische Annalen*, vol. 14 (1879), p. 557; Geodätische Linien und Enveloppen auf drei-axigen Flächen zweiten Grades, *ibid.* vol. 20 (1882), p. 557.

‖ Osgood, On the existence of a minimum of the integral $\int_{x_0}^{x_1} F(x, y, y')dx$ when x_0 and x_1 are conjugate points, *Transactions Amer. Math. Soc.*, vol. 2 (1901), p. 166. See also Kneser, *Variationsrechnung*, p. 97.

¶ Lectures on the Calculus of Variations at the University of Berlin, 1879.

revolves about a and approaches ab, its intersection with the latter approaches a' as a limit.

In the following paragraphs the analytical method employed in determining the geodesic lines and conjugate points is that of the calculus of variations as developed by Weierstrass.* The results are here briefly stated. Suppose a curve C_0,

$$x = \phi(u), \qquad y = \psi(u),$$

and an integral,

$$I = \int_{u_0}^{u_1} F(x, y, x', y')du,$$

taken along C_0 from $a(u = u_0)$ to $b(u = u_1)$. If I is to be independent of the parametric representation of the curve, then F must satisfy the functional relation†

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y'), \qquad \kappa > 0.$$

From this equation it can be shown that a function F_1 exists defined as follows:

$$F_1 = \frac{1}{y'^2}F_{x'x'} = \frac{-1}{x'y'}F_{x'y'} = \frac{1}{x'^2}F_{y_1y_1}.‡$$

Suppose, as is the case in most examples,

I $$F_1 > 0 \quad \text{along the arc } ab.$$

Then C_0 may make I a minimum, but if it does, it must satisfy also Legendre's condition,

II $$G_1 = F_x - \frac{d}{du}F_{x'} = 0, \qquad G_2 = F_y - \frac{d}{du}F_{y'} = 0.$$

The equations $G_1 = 0$ and $G_2 = 0$ are not independent, for it can be shown that§

$$x'G_1 + y'G_2 = 0.$$

Either equation is of the second order. Kneser‖ has shown that it is possi-

* *l. c.* See also Kneser, *Variationsrechnung;* and Osgood, Sufficient conditions in the Calculus of Variations, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1901), p. 105.

† Kneser, *l. c.*, p. 7.

‡ Literal subscripts denote partial differentiation.

§ Kneser, *l. c.*, p. 10.

‖ *l. c.*, §§ 29, 31.

ble to find a system of curves which are solutions of the equations II, and which have the form,

$$x = \phi(u, \alpha, \beta), \qquad y = \psi(u, \alpha, \beta),$$

where α and β are arbitrary constants. Suppose that C_0 is the one of these solutions defined by $\alpha = \alpha_0$, $\beta = \beta_0$. Then the arc ab will make I a minimum if besides I and II the following condition is also satisfied:

III $$\Theta(u, u_0) = \begin{vmatrix} \theta_1(u) & \theta_1(u_0) \\ \theta_2(u) & \theta_2(u_0) \end{vmatrix} \neq 0 \quad \text{for } u_0 < u \leqq u_1,$$

where

$$\theta_1(u) = \begin{vmatrix} x_\alpha & x' \\ y_\alpha & y' \end{vmatrix}, \qquad \theta_2(u) = \begin{vmatrix} x_\beta & x' \\ y_\beta & y' \end{vmatrix},$$

and in the derivatives x_α, x', etc., $\alpha = \alpha_0$, $\beta = \beta_0$. The conjugate point a' is the point on C_0 where condition III ceases to hold, that is, where $\Theta(u, u_0)$ first becomes zero. Conditions I, II and III are sufficient conditions for a minimum, but not necessary.

If the equations of the anchor ring are in the form

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi), \qquad z = z(\phi, \psi),$$

the integral expressing the length of the arc of any curve

$$\phi = \phi(u), \quad \psi = \psi(u)$$

will be

$$s = \int_{u_0}^{u_1} \sqrt{E\phi'^2 + 2F\phi'\psi' + G\psi'^2}\, du$$

where E, F, and G have the values

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2, \qquad G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2,$$

$$F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \psi}.$$

The geodesic lines are the curves upon the surface which make this integral a minimum. The object of the present paper is therefore threefold: the discussion of (1) the function F_1 for the integral s, (2) the curves which satisfy

$G_1 = 0$ or $G_2 = 0$ for this integral, and (3) the conjugate points defined by the zeros of $\Theta(u, u_0)$.

2. The Equations of the Geodesic Lines on the Anchor Ring.

The equations of the anchor ring can be written in either of the two following forms:

$$(1) \qquad \begin{cases} x = (a + b\cos\phi)\cos\psi = r\cos\psi, \\ y = (a + b\cos\phi)\sin\psi = r\sin\psi, \\ z = b\sin\phi \qquad\qquad = \sqrt{b^2 - (r-a)^2}, \end{cases}$$

where the quantities involved have the meanings indicated in figure 1. The integral s is

$$(2) \qquad s = \int_{u_0}^{u_1} F(\phi, \psi, \phi', \psi')\, du,$$

where

$$F(\phi, \psi, \phi', \psi') = \sqrt{b^2\phi'^2 + (a + b\cos\phi)^2\psi'^2}.$$

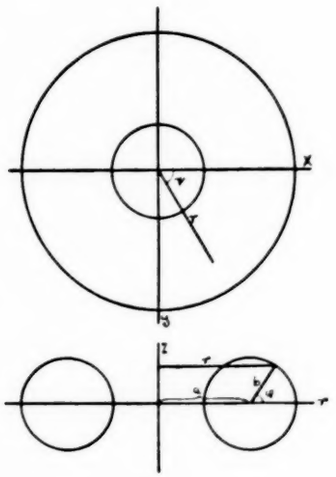


FIG. 1.

The function F_1 is found to be

$$F_1 = \frac{b^2(a + b\cos\phi)^2}{F^3},$$

which is always > 0. Condition I is therefore satisfied. After one integration the differential equation $G_2 = 0$ becomes

$$F_{\psi'} = \frac{(a + b\cos\phi)^2\,\psi'}{F} = \alpha. \tag{3}$$

The geodesic lines to be investigated are the solutions of this equation.

The constant of integration α has a simple geometrical meaning. It can always be taken positive, for a negative value in (3) only makes ψ decrease instead of increase with u. From the same equation

$$\alpha = r^2\frac{d\psi}{ds} = r\cos\chi, \tag{4}$$

where χ is the angle between the geodesic line G and the parallel circle C of radius r (figure 2). The last equation shows that α must always lie between 0 and $a + b$. The geodesic line can cut the inner equator ($r = a - b$) only if $\alpha \leqq a - b$, and if $\alpha > a - b$ the line touches and lies without the circle $r = \alpha$.

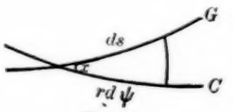


FIG. 2.

The equation (3) when solved for ψ' becomes

$$\psi'^2 = \frac{b^2\alpha^2\phi'^2}{(a + b\cos\phi)^2[(a + b\cos\phi)^2 - \alpha^2]} = \frac{b^2\alpha^2 r'^2}{r^2(r^2 - \alpha^2)[b^2 - (r - a)^2]}. \tag{5}$$

The complete integral with r as the parameter is

$$\begin{cases} \phi = \cos^{-1}\dfrac{r - a}{b}, \\ \psi = \displaystyle\int \frac{b\alpha\,dr}{r\sqrt{R(r)}} + \beta', \end{cases} \tag{6}$$

where

$$R(r) = (r^2 - \alpha^2)[b^2 - (r - a)^2],$$

and β' is an arbitrary constant.

It will be convenient for the discussion of the geodesic lines to have ϕ and ψ expressed in terms of Weierstrassian functions. For this purpose introduce the new independent variable u by means of the transformation,*

$$r = a + b + \frac{R'(a + b)}{4\left[\wp(u) - \frac{1}{24}R''(a + b)\right]}. \tag{7}$$

* Enneper *Elliptische Functionen*, 1890, p. 30.

Here $a + b$ is a root of $R(r) = 0$. If the other three roots are denoted by

$$a_\lambda = a - b, \quad a_\mu = \alpha, \quad a_\nu = -\alpha,$$

then equation (7) solved for $\wp(u)$ gives the following expressions for e_λ, e_μ, e_ν, and their differences:

$$(8)\begin{cases} e_\lambda = \frac{1}{24} R''(a+b) - \dfrac{R'(a+b)}{8b}, & e_\mu - e_\nu = b\alpha, \\ e_\mu = \frac{1}{24} R''(a+b) - \dfrac{R'(a+b)}{4(a+b-\alpha)}, & e_\nu - e_\lambda = \frac{1}{4}(a+b-\alpha)(-a+b-\alpha), \\ e_\nu = \frac{1}{24} R''(a+b) - \dfrac{R'(a+b)}{4(a+b+\alpha)}, & e_\lambda - e_\mu = \frac{1}{4}(a+b+\alpha)(a-b-\alpha). \end{cases}$$

Two cases must be distinguished according to the relative values of e_λ, e_μ, e_ν:

Case A: $0 < \alpha < a - b$: $e_\lambda > e_\mu > e_\nu$, $\therefore \lambda = 1, \mu = 2, \nu = 3$.
Case B: $a - b < \alpha < a + b$: $e_\mu > e_\lambda > e_\nu$, $\therefore \lambda = 2, \mu = 1, \nu = 3$.

Since

$$(9) \qquad \frac{dr}{du} = \sqrt{R(r)},$$

the integral expressing ψ is transformed by the substitution (7) into the form

$$\psi = \int \frac{b\alpha}{r}\, du + \beta'.$$

To find the value of this integral in terms of the Weierstrassian functions it is necessary first to express the integrand in the same manner. Two quantities r_1 and r_2 can be defined by the equations,

$$\wp(r_1) = \tfrac{1}{24} R''(a+b) - \frac{R'(a+b)}{4(a+b)},$$
$$\wp(r_2) = \tfrac{1}{24} R''(a+b).$$

Then

$$(10) \qquad r = (a+b)\,\frac{\wp(u) - \wp(r_1)}{\wp(u) - \wp(r_2)}.$$

The function $b\alpha/r$ is an elliptic function whose only poles are the points where $r = 0$, that is $u = \pm r_1 + 2m\omega_1 + 2m'\omega_3$, m and m' being any integers. The residues at $u = \pm r_1$ can be found by expansion. Since the values of dr/du in (9) at $+r_1$ and $-r_1$ differ only in sign, the developments of r at

these points begin as follows:

$$+ a\sqrt{a^2 - b^2}\,(u - v_1) + \cdot\cdot\cdot\cdot\cdot,$$
$$- a\sqrt{a^2 - b^2}\,(u + v_1) + \cdot\cdot\cdot\cdot\cdot.$$

The residues of ba/r are therefore

$$R_{v_1} = \frac{b}{\sqrt{a^2 - b^2}} = - R_{-v_1}.$$

Any elliptic function can be expressed in terms of Weierstrassian functions when its poles and their principal parts* are known. In the present case the difference

$$\frac{ba}{r} - \frac{b}{\sqrt{a^2 - b^2}}\Big[\zeta(u - v_1) - \zeta(u + v_1)\Big]$$

is an elliptic function with no poles, and by a well-known theorem† must therefore be a constant. The value of the constant is found by putting $u = 0$, which makes $\wp(u)$ infinite and $r = a + b$. Then

$$\frac{ba}{r} = \frac{b}{\sqrt{a^2 - b^2}}\Big[\zeta(u - v_1) - \zeta(u + v_1) + 2\zeta(v_1)\Big] + \frac{ba}{a + b}.$$

By a simple integration the equations (6) become

$$(11)\qquad \begin{cases} \phi = \cos^{-1}\dfrac{r - a}{b}, \\ \psi = \displaystyle\int_\beta^u \frac{ba}{r}\,du = G(u) - G(\beta), \end{cases}$$

where

$$(12)\qquad G(u) = \frac{b}{\sqrt{a^2 - b^2}}\Big[\log\frac{\sigma(v_1 - u)}{\sigma(v_1 + u)} + 2u\zeta(v_1)\Big] + \frac{ba}{a + b}\,u.$$

When the value of r from (10) is substituted, these equations have the form

$$\phi = \phi(u, a, \beta), \qquad \psi = \psi(u, a, \beta),$$

a being involved explicitly and through the branch points of $y^2 = R(r)$ from which the $\wp$-, ζ-, and σ-functions are determined. They also satisfy the differential equation $G_2 = 0$. Therefore *equations* (11) *represent a doubly infinite system of geodesic lines on the anchor ring.*

---

* Hauptteil, *cf. Encyclopædie* II, B. 1, §9.

† Harkness and Morley, *Introduction to the Theory of Analytic Functions*, 1898, p. 255

3. The Geometrical Character of the Geodesic Lines. It is proposed to show that the geodesic lines on the anchor ring are the meridian circles, the two equators, or curves of the types shown in figures 3–5. The solutions of (3) for certain values of a are simple. For example, when $a = 0$ or $a + b$, the only solutions are the meridian circles and the outer equator. When $a = a - b$ the inner equator is a solution for which ϕ remains constant. It remains then to discuss the geodesic lines for Case A, Case B, and the lines for $a = a - b$ along which ϕ is variable.

The position of a point upon the anchor ring is completely determined as soon as r, ψ, and z are known. The expression defining z for a point upon a geodesic line is found from (1) and (10),

$$z = \pm \sqrt{2b(a+b)\left[\wp(r_1) - \wp(r_2)\right]}\ \frac{\sqrt{\wp(u) - e_\lambda}}{\wp(u) - \wp(r_2)},$$

or since*

$$\sqrt{\wp(u) - e_\lambda} = \frac{\sigma_\lambda(u)}{\sigma(u)},$$

$$(13) \qquad z = \pm \sqrt{2b(a+b)\left[\wp(r_1) - \wp(r_2)\right]}\ \frac{\sigma_\lambda(u)}{\sigma(u)\left[\wp(u) - \wp(r_2)\right]}.$$

Consider now the values of r, ψ, and z in (10), (11), and (13). In the first place they represent similar curves when a is the same but β has different values, for a change in β does not alter r or z, and only adds a constant to ψ. *The form of the geodesic line depends therefore only upon the value of* a. *If* a *is fixed, different values of* β *define curves which can be made to coincide by rotations about the z-axis.*

From the equations†

$$\sigma_1(u + 2\omega_1) = -e^{2\eta_1(u+\omega_1)}\sigma_1(u), \qquad \sigma(u + 2\omega_1) = -e^{2\eta_1(u+\omega_1)}\sigma(u),$$
$$\wp(u + 2\omega_1) = \wp(u),$$

it follows that for curves of Case A, r and z are periodic with the period $2\omega_1$. Since also

$$(14) \qquad \sigma(-u) = -\sigma(u),$$

* Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Funktionen*, p. 21.
† Schwarz, *l. c.*, p. 22.

equations (11) and (12) show that

$$\psi(u + 2\omega_1) = \psi(u) + 2\Psi,$$

where 2Ψ is a constant:

$$2\Psi = G(2\omega_1). \tag{15}$$

Any geodesic line of Case A consists of repetitions of the part for which $-\omega_1 \leqq u \leqq \omega_1$, *and two similar parts can be made to coincide by rotations about the z-axis.*

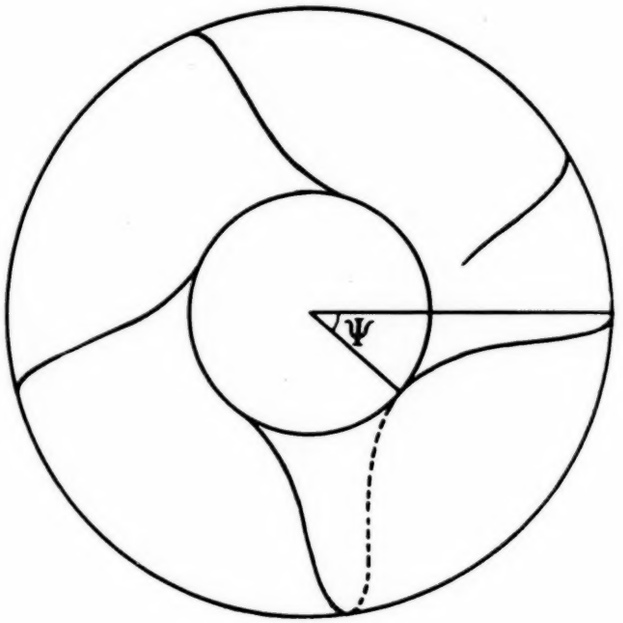


FIG. 3.

Since the form of the line depends only upon a, suppose $\beta = 0$. From (14) and the equations

$$\zeta(-u) = -\zeta(u), \qquad \wp(-u) = \wp(u),$$

it follows that

$$r(-u) = r(u), \qquad \psi(-u) = -\psi(u), \qquad z(-u) = -z(u).$$

The curve for $-\omega_1 \leqq u \leqq 0$ is the reflection of the part for $0 \leqq u \leqq \omega_1$, first through the xz-plane and then through the xy-plane.

Since*

$$\wp(\omega_1) = e_1,$$

r has the values $a \pm b$ for $u = 0$ and $u = \omega_1$ respectively. Between these two values of u, $\psi' = \dfrac{ha}{r}$ is positive, and since r must diminish near $u = 0$, the expression (9) for dr/du must be negative. *In Case A the geodesic line winds about the ring crossing the two equators alternately. The parts between any*

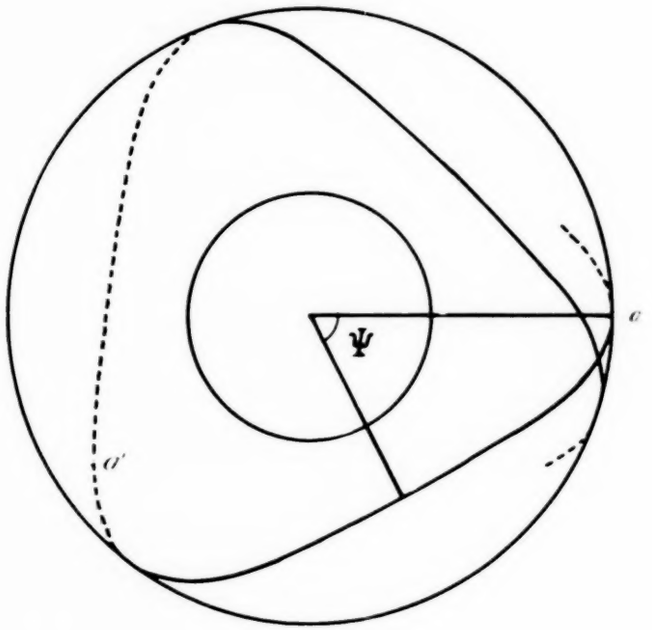


FIG. 4.

two successive intersections with either equator are similar and can be made to coincide by rotations about the z-axis (see figure 3). The two lines for the opposite values of z in (13) have the same shape, but wind about the ring in opposite directions.

The difference in the discussion for Case *B* arises because λ and μ are

* Schwarz, *l. c.*, p. 12.

interchanged. The equation

$$\sigma_2(u + 2\omega_1) = e^{2\eta_1(u+\omega_1)}\,\sigma_2(u)$$

shows that z has the period $4\omega_1$ instead of $2\omega_1$, and the part of the curve for $-2\omega_1 \leqq u \leqq 0$ is the reflection of the part for $0 \leqq u \leqq 2\omega_1$. In the intervals $0 \leqq u \leqq \omega_1$ and $\omega_1 \leqq u \leqq 2\omega_1$, dr/du has opposite signs, and r varies

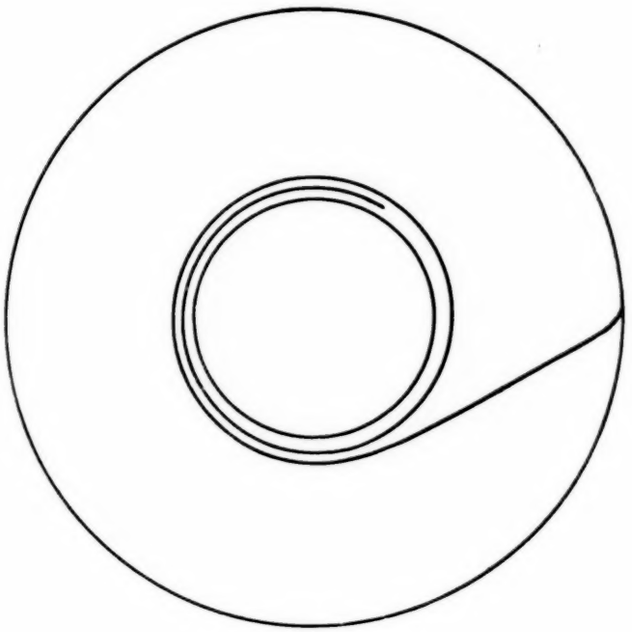


FIG. 5.

between the values $a + b$ and a. So that *in Case B the geodesic line winds back and forth across the outer equator and is tangent alternately to the parallel circles* $r = a$, $z > 0$, *and* $r = a$, $z < 0$. *The parts between successive contacts with either of these circles are similar, and can be made to coincide by rotations about the z-axis* (see figure 4). The two curves for the opposite values of z can in this case be made to coincide.

When $a = a - b$ in (6), a curve is found for which ϕ is not a constant. The integral can be expressed by means of logarithms. Substitute

$$r = (a - b)\,\frac{a + b - bu^2}{a - b + bu^2}.$$

Then

$$\psi = \sqrt{\frac{b}{a+b}}\,\log\sqrt{\frac{1+u}{1-u}} - \frac{b}{a+b}\,\log\frac{\sqrt{a+b}+u\sqrt{b}}{\sqrt{a+b}-u\sqrt{b}} + C,$$

$$z = \pm\, 2b\sqrt{a+b}\,\frac{u\sqrt{1-u^2}}{a-b+bu^2},$$

where C is an arbitrary constant. These equations have properties similar to those of Case A1. When $C = 0$ and u increases from 0 to 1, r decreases from $a + b$ to $a - b$, and ψ increases from 0 to ∞. The curve for negative values of u is the reflection of the positive part. Therefore *when $a = a - b$, equations (6) represent a geodesic line which cuts the outer equator in only one point. Proceeding in opposite directions from this point the curve lies on opposite sides of the xy-plane and approaches the inner equator asymptotically from above and below* (see figure 5). As in Case A1 the lines for opposite values of z wind in opposite directions, and can not be made to coincide.

4. Relation Between the Geodesic Lines for Different Values of the Constant a. The changes which the geodesic line undergoes as a varies from 0 to $a + b$ can be found by forming the derivative Ψ_a. From (15)

$$2\Psi_a = G_a(2\omega_1) + 2G_u(2\omega_1)\,\frac{d\omega_1}{da}.$$

If r is expanded by Taylor's formula at the point $u = 2\omega_1$ by means of (9), and the derivatives r_a and r_u taken, it is found that

$$2\,\frac{d\omega_1}{da} = -\left[\frac{r_a}{r_u}\right]_{u=2\omega_1}.$$

The value of Ψ_a is then

$$2\Psi_a = H(2\omega_1),$$

where

$$H(u) = \frac{1}{r_u}\begin{vmatrix} G_a & G_u \\ r_a & r_u \end{vmatrix}.$$

The function $H(u)$ is useful in forming $\Theta(u, u_0)$ as well as Ψ_a. It can be calculated in terms of the Weierstrassian functions from the properties of its derivative, which is expressible in terms of r. Since from (11)

$$G' = \frac{ba}{r},$$

it follows that

$$H'(u) = \frac{b}{r}\left\{1 - a\frac{d}{du}\frac{r_a}{r_u}\right\}.$$

Consider the functions $\frac{1}{2}\left\{\frac{1}{r+a} - \frac{1}{r-a}\right\}$ and $\frac{d}{du}\frac{r_a}{r_u}$. The former is an elliptic function whose poles are at $u = \omega_\mu$ and $u = \omega_\nu$, where r has the values a and $-a$ respectively. The expansions can be found by means of (9). It turns out that the only negative powers are of order two, and these have the coefficients

$$R_\mu = \frac{-2}{R'(a)}, \qquad R_\nu = \frac{+2}{R'(-a)}. \tag{16}$$

When $u = 0$ the function has the value $\frac{2ba}{R'(a+b)}$.

But these are exactly the properties of $\frac{d}{du}\frac{r_a}{r_u}$. From (10)

$$\frac{r_a}{r_u} = \frac{1}{\varphi'(u)}\frac{\partial\varphi(u)}{\partial a} + C_1\frac{\varphi(u)}{\varphi'(u)} + \frac{C_2}{\varphi'(u)}, \tag{17}$$

where C_1 and C_2 are constants. a is involved in $\varphi(u)$ through the branch points $a_\mu = a, a_\nu = -a$, so that

$$\frac{\partial\varphi(u)}{\partial a} = \frac{\partial\varphi(u)}{\partial a_\mu} - \frac{\partial\varphi(u)}{\partial a_\nu}.$$

A formula for the derivation of $\sigma(u)$ with re pect to a branch point a has been derived by Bolza:*

$$R'(a)\frac{\partial\sigma}{\partial a} = \tfrac{1}{12}G_2u^2\sigma + \tfrac{1}{12}R''(a)\left[u\frac{\partial\sigma}{\partial u} - \sigma\right] + \frac{\partial^2\sigma}{\partial u^2}.$$

* The elliptic σ-functions considered as a special case of the hyperelliptic σ-functions. *Transactions Amer. Math. Soc.*, vol. 1, 1900, p. 63.

One for $\wp(u)$ can be found by dividing this by $\sigma(u)$, differentiating twice, and using the formulae*

$$\frac{\sigma''}{\sigma} = \zeta^2 - \wp, \qquad \wp'' = 6\wp^2 - \frac{g_2}{2}.$$

Then

$$R'(a)\frac{\partial \wp}{\partial u} = \tfrac{1}{12} R''(a)\left[2\wp + u\wp'\right] + 2\zeta\wp' + 4\wp^2 - \tfrac{2}{3}g_2. \tag{18}$$

From (17) and (18) it is seen that $\frac{d}{du}\frac{r_a}{r_u}$ is an elliptic function with poles possible at $\omega_\lambda, \omega_\mu, \omega_\nu$, where $\wp'(u)$ vanishes. At these points r has the values $a-b, a, -a$ respectively, and can be expanded in powers of $u-\omega_\rho$ ($\rho = \lambda, \mu, \nu$) by means of (9). The quotient r_a/r_u is then found to have poles at ω_μ and ω_ν, but none at ω_λ: and its derivative has therefore poles at ω_μ and ω_ν, with the same principal parts R_μ, R_ν, as in (16). Similarly the derivatives of the expansion of r at $u=0$ give $\frac{d}{du}\frac{r_a}{r_u}$ the same value $\frac{2ba}{R'(a+b)}$ when $u=0$.

The difference between the two functions just considered is an elliptic function with no poles, and therefore a constant. The constant is found to be zero by putting $u=0$, so that the two are equal. By using this result

$$H'(u) = \frac{b}{2}\left[\frac{1}{r-a} + \frac{1}{r+a}\right].$$

A comparison of the principal parts at poles and of expansions at $u=0$ as before, shows that

$$\frac{1}{b}H'(u) = -R_\mu\left[\wp(u-\omega_\mu) + e_\lambda\right] + R_\nu\left[\wp(u-\omega_\nu) + e_\lambda\right].$$

If r and G are expanded at $u=0$ by Taylor's formula with the help of (9) and the equation $G' = ba/r$ respectively, and the derivatives r_a and G_a then taken, it is found that $H(0) = 0$. The equation for $H'(u)$ therefore gives, when integrated,

$$\frac{1}{b}H(u) = R_\mu\left[\zeta(u-\omega_\mu) + \eta_\mu + e_\lambda u\right] - R_\nu\left[\zeta(u-\omega_\nu) + \eta_\nu + e_\lambda u\right].$$

* Schwarz, *l. c.*, p. 12.

From this result the expression for Ψ_a is

$$\Psi_a = \tfrac{1}{2} H(2\omega_1) = b(R_\mu - R_\nu)(\eta_1 + e_\lambda \omega_1),$$

since $\zeta(u)$ has the property*

$$\zeta(u + 2\omega_\rho) = \zeta(u) + 2\eta_\rho, \qquad \rho = \lambda, \mu, \nu.$$

The sign of Ψ_a can be determined. From (16)

$$R_\mu - R_\nu > 0 \text{ for Case } A; \qquad R_\mu - R_\nu < 0 \text{ for Case } B.$$

In Case A*

$$\eta_1 + e_1\omega_1 = \frac{\pi^2}{2\omega_1}\left\{\tfrac{1}{2} + \Sigma_n \frac{4h^{2n}}{(1+h^{2n})^2}\right\} > 0,$$

where $h = e^{\pi i \omega_3/\omega_1}$ is a real positive quantity; and in Case B,

$$\eta_1 + e_2\omega_1 = \frac{\pi^2}{2\omega_1} \Sigma_n \frac{4h^{2n-1}}{(1+h^{2n-1})^2} > 0.$$

Ψ therefore increases as a increases from 0 to $a - b$, and decreases as a increases from $a - b$ to $a + b$.

The limits of Ψ in Case A are 0 and ∞, and in Case B, ∞ and $\frac{\pi}{2}\sqrt{\frac{b}{a+b}}$.

For in either case†

$$\omega_1 = \frac{1}{\sqrt{e_1 - e_3}} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \qquad \kappa^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

From equations (8) the limits of $e_1 - e_3$ and κ can be found. Then

$$\lim_{a=0} \omega_1 = \frac{2}{\sqrt{a^2 - b^2}} \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{\sqrt{a^2 - b^2}},$$

and similarly

$$\lim_{a=a-b} \omega_1 = \infty, \qquad \lim_{a=a+b} \omega_1 = \frac{\pi}{2\sqrt{b(a+b)}}.$$

The integral in (11) shows that

$$\lim_{a=0} \Psi = 0, \qquad \lim_{a=a-b} \Psi = \infty, \qquad \lim_{a=a+b} \Psi = \frac{\pi}{2}\sqrt{\frac{b}{a+b}}.$$

For Case A and a near to zero the geodesic line is therefore only a little

* Schwarz, *l. c.*, p. 9.

† Schwarz, *l. c.*, pp. 36 and 32.

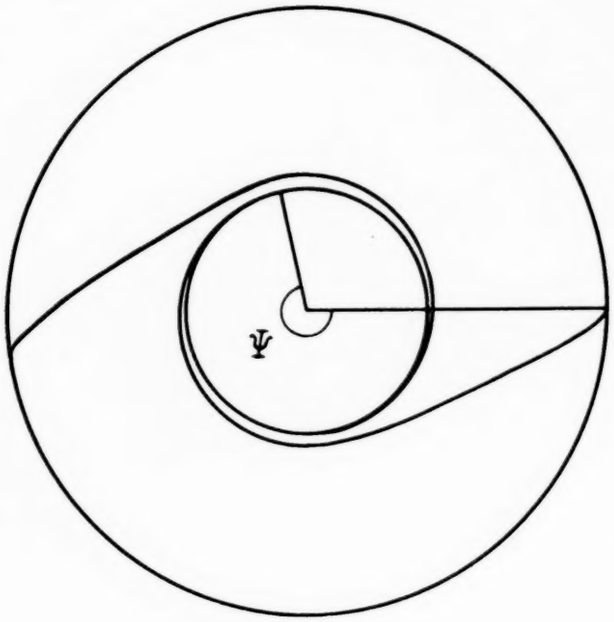


FIG. 6.

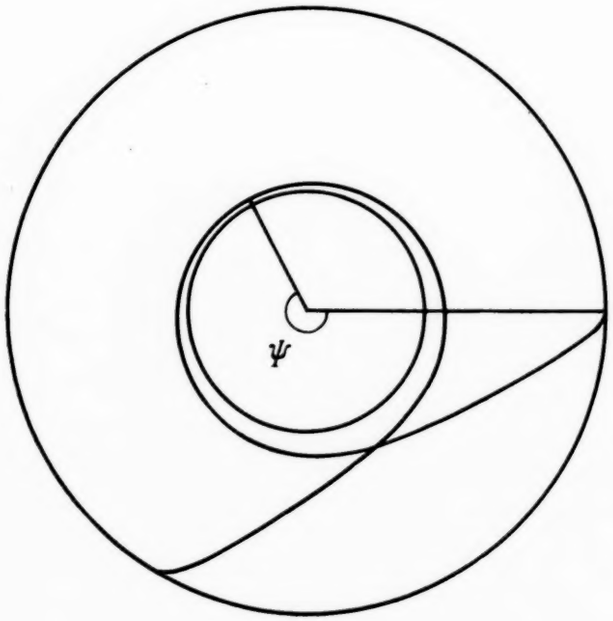


FIG. 7.

different from a meridian. As a increases it takes first the form of figure 3, then that of figure 6, and finally if a is near to $a - b$ winds many times around the inner equator before crossing it. For Case B and a near to $a - b$, the geodesic line winds many times around the inner equator before touching the circle $r = a$. As a increases it takes successively the forms shown in figures 4 and 7. When a is near to $a + b$ the line almost coincides with the outer equator, and intersects it in two points whose radii make an angle a little greater than $\pi\sqrt{\frac{b}{a+b}}$ with each other. The line of figure 5 for $a = a - b$ is a limiting line between those of Cases A and B.

5. Determination of Conjugate Points. The conjugates to a point $a(u = u_0)$ upon a geodesic line are determined by the zeros of $\Theta(u, u_0)$ which lie nearest to u_0. The functions $\theta_1(u)$ and $\theta_2(u)$ of §1 are, from (11),

$$(19)\qquad \begin{cases} \theta_1(u) = \dfrac{r_u}{b \sin\phi}\left\{H(u) - G_a(\beta)\right\}, \\ \theta_2(u) = -\dfrac{a}{r(\beta)}\dfrac{r_u}{\sin\phi}. \end{cases}$$

Since from (9)

$$r_u = b \sin\phi \sqrt{r^2 - a^2},$$

the value of $\Theta(u, u_0)$ is

$$(20)\qquad \Theta(u, u_0) = C_3\sqrt{r^2 - a^2}\,[H(u) - H(u_0)],$$

C_3 being a constant. The real zeros of this function are to be investigated.

For a point on a line of Case A, $\Theta(u, u_0)$ has only the zero $u = u_0$. The factor $\sqrt{r^2 - a^2}$ is never zero, for r is always greater than a. $H(u) - H(u_0)$ vanishes at $u = u_0$, is finite for all real values of u, and has a positive derivative. *Therefore no point on a geodesic line of Case A can have a conjugate.*

When the geodesic line is one of case B and tangent to the circle $r = a$ at the point a, $r(u_0)$ has the value a, and $\left[\dfrac{dr}{du}\right]_{u=u_0}$ therefore vanishes. In this special case the expression for $\Theta(u, u_0)$, when calculated from the functions (19), is simply $C_4\sqrt{r^2 - a^2}$, where C_4 is a constant. *The conjugate points to a point of tangency of a geodesic line of Case B with the circle $r = a$, $z > 0$, are therefore the two nearest points of tangency with the circle $r = a$, $z < 0$.*

For other values of u_0 in Case B, $\Theta(u, u_0)$ in (20) has no zeros due to the

factor $\sqrt{r^2 - a^2}$. For, from the properties of the $\wp$-function in (7), it follows that the only real values of u which make $r = a$ are $u = (2k + 1)\omega_1$, where k is any integer. At such points $\sqrt{r^2 - a_2}$ has a zero of order one and $H(u)$ has a pole of order one, so that $\Theta(u, u_0)$ is finite and does not vanish. The factor $H(u) - H(u_0)$ has a zero in every interval from $(2k - 1)\omega_1$ to $(2k + 1)\omega_1$. For at either of these values $H(u)$ is infinite, and at other points its derivative is positive. In any such interval therefore $H(u)$ increases from $-\infty$ to $+\infty$ and takes the value $H(u_0)$ once. The geodesic line can be divided into sections between consecutive points of tangency with the circles $r = a$, $z > 0$ and $r = a$, $z < 0$. *Any point on a certain section of a line of Case B has two conjugates, one in each adjacent section.*

As u_0 traverses its interval $(2k - 1)\omega_1$ to $(2k + 1)\omega_1$, the value u' making

$$H(u') = H(u_0) \tag{21}$$

traverses its interval. For if u' is regarded as a function of u satisfying (21), its derivative is positive, and in order to make $H(u')$ take all possible values $H(u_0)$, u' must traverse its whole interval. Therefore *if the point a moves from one end of a section of the geodesic line to the other, its conjugates traverse the two adjacent sections in the same direction.*

It is interesting to notice that the radii of a and its conjugate a' make an angle with each other which is never less than 2Ψ. This follows from the property,

$$H(u + 2\omega_1) = H(u) + H(2\omega_1),$$

where $H(2\omega_1) < 0$ as in §4. The value of $H(u_0 + 2\omega_1)$ is less than $H(u_0)$, so that u' must lie beyond $u_0 + 2\omega_1$ in order to satisfy (21). As an illustration, a point upon the outer equator will have a conjugate beyond the next intersection of the geodesic line with that equator, as indicated in figure 4. The conjugate point marks the place where the geodesic line ceases to be a relative minimum. There may be a curve on the surface joining a with some point b between a and a', and which is shorter than the corresponding arc of the geodesic line, but a neighborhood of ab, perhaps necessarily small, can always be found in which no such curve exists.

The conjugates of a point a upon either of the two equators, or a meridian, or the line (6) for $\alpha = a - b$, cannot be conveniently determined from

$\Theta(u, u_0)$ as expressed in (20). In these cases the elliptic functions degenerate. But the conjugates can be determined by means of the equation*

$$\frac{d^2m}{ds^2} + km = 0, \tag{22}$$

where k is the curvature of the surface and s the arc of the geodesic line measured from a. If m is an integral of (22) which vanishes for $s = 0$, the conjugates to a are determined by the values of s nearest to zero which make m vanish. The equation (22) for geodesic lines is the transform of one which has $\Theta(u, u_0)$ as an integral in the problem of the Calculus of Variations. m differs from $\Theta(u, u_0)$ only by a factor which never becomes zero.

Upon the anchor ring

$$k = \frac{r - a}{br}.$$

From (2) and (5),

$$\frac{ds}{dr} = \frac{ds}{d\phi}\frac{d\phi}{dr} = -\frac{br}{\sqrt{R(r)}}.$$

Equation (22) has therefore the integral

$$m = \sqrt{r^2 - a^2}\int_0^s \frac{ds}{r^2 - a^2},$$

which is indeed $\Theta(u, u_0)$ in different form and multiplied by a constant. When a is 0 or $a - b$, m has only the zero $s = 0$. *Therefore a point on a meridian circle, or on the curve in figure* 5, *can have no conjugate.*

For the inner and outer equators k has the values,

$$k = \mp\frac{1}{b(a \mp b)},$$

with the upper and lower signs respectively. Equation (22) shows that for the inner equator m, $\frac{dm}{ds}$, and $\frac{d^2m}{ds^2}$ have the same sign, so that if $m = 0$ for $s = 0$, it cannot vanish for any other value of s. For the outer equator, (22) has the integral

$$m = \sin\frac{s}{\sqrt{b(a + b)}} = \sin\psi\sqrt{\frac{a + b}{b}},$$

* Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, p. 97.

and the first values of ψ for which this vanishes are

$$\psi = \pm \pi \sqrt{\frac{b}{a+b}}.$$

These results show that *no point on the inner equator has a conjugate. But every point on the outer equator has two conjugates whose radii make the angles* $\pm \pi \sqrt{\frac{b}{a+b}}$ *with the radius of the given point.* The latter result might have been deduced in §4 from the fact that the intersections of the geodesic lines through a with the outer equator, have a limiting point whose radius makes the angle $\pi \sqrt{\frac{b}{a+b}}$ with that of a.

From the foregoing paragraphs it follows that *a geodesic line on the anchor ring which has points in common with the inner equator or approaches it asymptotically, is a relative minimum between any two of its points. But any point on a geodesic line which does not touch the inner equator has two conjugates.*

6. Classification of the Points upon the Anchor Ring. A point can be classified according to Mangoldt's scheme by finding out all of the geodesic lines which pass through it. If the given point has a conjugate on any one of them, it cannot be of the first kind. From the results of §5, any point on the inner equator is of the first kind because the geodesic lines through it can have no conjugates.

Through any other point (r_0, ψ_0, z_0) of the anchor ring there are two geodesic lines corresponding to each value of α less than r_0, and symmetrical with respect to the plane of the meridian $\psi = \psi_0$. This admits of an analytical proof, but is evident geometrically. For any two lines of Case A (or for $\alpha = a - b$, see figure 5) which have the same value of α but wind in opposite directions about the ring, cut the circle $r = r_0$, $z = z_0$ at angles χ and $-\chi$ respectively; and two such intersections can be made to coincide with the given point by rotations about the z-axis. Similarly any line of Case B which has $\alpha < r_0$ cuts the circle $r = r_0$, $z = z_0$ at two different angles, and by rotations can be made to pass through the given point in two directions symmetrical with respect to the meridian $\psi = \psi$. The two lines passing through the given point and corresponding to $\alpha = a - b$ are limiting lines between those of Cases A and B (see figure 8). From (4) there are no lines through the given point

for $a > r_0$. Since the lines of Case B and the outer equator are the only ones which have conjugate points, the following theorem can be stated:

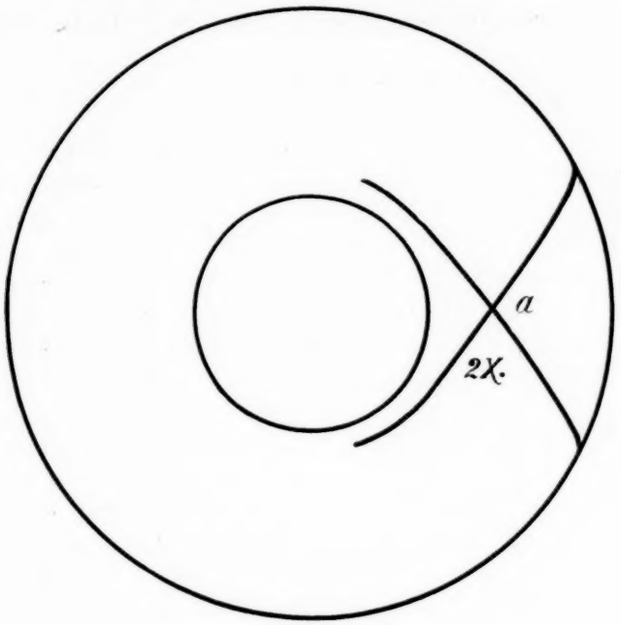


FIG. 8.

The points on the inner equator of the anchor ring are of the first kind according to Mangoldt's classification, and all others are of the second kind. The only geodesic lines on which there are conjugates to a point (r_0, ψ_0, z_0) *of the second kind are those which pass through that point and make an angle less than* χ_0 *with the parallel circle* $r = r_0$, $z = z_0$ *where*

$$\cos \chi_0 = \frac{a - b}{r_0} .$$

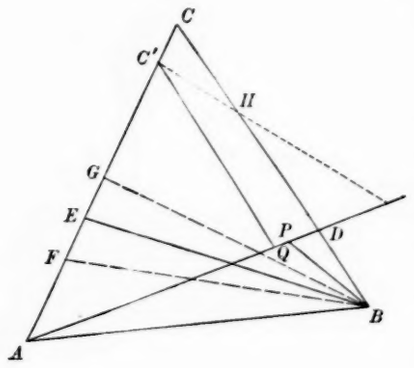
THE UNIVERSITY OF MINNESOTA,

MAY, 1902.

PROOF OF A THEOREM CONCERNING ISOSCELES TRIANGLES*

By H. F. Blichfeldt

The theorem " a triangle is isosceles if the bisectors of two of its angles are equal," has attracted considerable attention, owing to the seeming difficulty of its proof. The writer has discovered a proof that covers the case of the internal trisectors, etc., as well as the bisectors, and holds as well for the so-called non-euclidean as for the euclidean space; *i. e.* it does not depend upon parallel lines, nor on the angle-sum of a triangle being any given amount. Although numerous proofs have been given in various books and periodicals,† the writer submits herewith the proof referred to, as it may possess some interest on account of its somewhat more general character.



The theorem may be stated thus:

In a triangle ABC, straight lines AD and BE are drawn, intersecting the sides BC and AC respectively in the points D and E, and dividing the angles CAB and CBA internally in the same ratio, so that

angle CAD: angle DAB = angle CBE: angle EBA;

then, if BE and AD are equal, the triangle ABC is isosceles.

This theorem is true whether the triangle be drawn in euclidean or in noneuclidean space, provided it can be drawn entirely within a certain limited region of the space considered.

---

* This article is a modification of part of a paper: Demonstration of a Pair of Theorems in Geometry, by the same author, read before the Edinburgh Math. Soc., Dec. 13, 1901

† *e. g.* in G. B. Halsted's *Elementary Synthetic Geometry*, pp. 44–45. Some references are given in J. S. Mackay's *The Elements of Euclid*, Edin., 1888, p. 109.

The proof will be based on the following propositions only, — aside from the pan-geometric axioms as to superposition, etc.*

Within a certain limited region of space,

1. Two triangles are congruent if a side and two adjacent angles of one are respectively equal to a side and two adjacent angles of the other.
2. The exterior angle of a triangle is greater than either opposite interior angle.
3. If two sides of a triangle are unequal, opposite the greater side lies the greater angle.
4. One, and only one, perpendicular can be drawn from any given point to a given straight line.
5. If oblique lines drawn from a point in a perpendicular to a straight line cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

To prove our theorem it is sufficient to prove that if the triangle ABC is not isosceles, the lines BE and AD are unequal. This we shall proceed to do.

Given, in triangle ABC,

$$AC > BC,$$

$$\text{angle } CAD : \text{angle } DAB = \text{angle } CBE : \text{angle } EBA;$$

to prove $AD > BE$.

Since $AC > BC$, angle $B >$ angle A (Prop. 3). Hence:

$$\text{angle } CBE > \text{angle } CAD, \qquad \text{angle } EBA > \text{angle } DAB.$$

Draw BG and BF making the angles CBG and FBA respectively equal to the angles CAD and DAB, and meeting AC in the points G and F respectively.

---

* See H. P. Manning, *Non Euclidean Geometry*, for the proof of these propositions, and for the immediate assumptions underlying them. In chapter 1 are given all the propositions referred to. It may be noticed that the proof holds if the triangle be drawn on a certain limited portion of any surface of constant curvature — the sphere for instance — in ordinary euclidean space. By "straight line" we must then understand "geodesic," and the word "congruent" must be replaced by the phrase "congruent or symmetric" in proposition 1, and by the word "symmetric" wherever it occurs in the subsequent demonstration. The term "symmetric" is used here in the sense in which it is generally used in spherical geometry.

α. To prove $BF < AD$.

Locate a point P in AD so that angle PBA = angle A. P lies between A and D, as angle $B >$ angle A. Hence $AP < AD$. Now, the triangles APB and BFA are congruent (Prop. 1). Hence, $BF = AP < AD$.

β. Again, to prove $BG < AD$.

Locate C' on AC so that $AC' = BC$. The point C' lies between A and C, as $AC > BC$. Draw $C'Q$ making angle $AC'Q$ = angle ACD. The lines $C'Q$ and AD will meet at Q, say, forming a triangle $AC'Q$ congruent with the triangle BCG (Prop. 1).

The lines $C'Q$ and CD, extended if need be, cannot intersect in the limited region of the plane considered. If they met at H, say, a point within the region considered, we should have in this region a triangle $HC'C$ having an external angle, $AC'H$ equal to an opposite interior angle, ACH, which is impossible by Prop. 2.

Thus Q must lie between A and D, so that $AQ < AD$. But $AQ = BG$, as the triangles $AC'Q$ and BCG are congruent. Hence, $BG = AQ < AD$.

γ. Finally, to prove BE either less than BF or less than BG.

Drop the perpendicular from B upon the line AC (Prop. 4). Then, one at least of the oblique lines BG and BF will cut off a greater distance from the foot of the perpendicular on the line AC than will BE. Hence, by Prop. 5, BE is either less than BF or less than BG.

By (α) and (β), each of the latter is less than AD. Hence, $AD > BE$.

STANFORD UNIVERSITY,
APRIL, 1902.

AN ELEMENTARY EXPOSITION OF FROBENIUS'S THEORY OF GROUP-CHARACTERS AND GROUP-DETERMINANTS

By Leonard Eugene Dickson

Introduction

In a series of recent memoirs* in the *Berliner Sitzungsberichte*, Frobenius has developed an elaborate theory which has already attracted considerable attention, both on account of the beauty of the results and of its importance in various applications. For example, it furnishes a determination of the minimum number of variables upon which a given group can be represented as a linear group (Klein's normal problem). After the success which had attended the introduction of characters of a commutative group, particularly in the determination of all its subgroups,† it was a natural step for Frobenius to introduce characters of any finite group.

Burnside‡ has re-established the main results due to Frobenius and has obtained new results on certain continuous groups related to the theory (see §3 below). A simplification in the method and a generalization of the results have been made by the writer.§ The papers just cited naturally rely upon technical group theory.

The present paper presents the chief results due to Frobenius and follows his methods as closely as an elementary treatment will admit. Symbolic notations are avoided and the material is presented with considerable detail, a number of illustrative examples being worked out. The paper is practically self-contained, involving no dependence upon technical branches of mathematics. Of group-theory itself, only the most elementary knowledge is pre-

* Über Gruppencharaktere. 1896, pp. 985–1021; Über die Primfactoren der Gruppendeterminante, 1896, pp. 1343–1382; Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, 1897, pp. 994–1015; 1899, p. 482; Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, 1898, pp. 501–515; Über die Composition der Charaktere einer Gruppe, 1899, pp. 330–339.

† Compare Weber's excellent treatment, *Algebra*, II, Zweiter Abschnitt, where references are given.

‡ *Proc. Lond. Math. Soc.*, vol. 29 (1898), pp. 207–224, 546–565.

§ *Trans. Amer. Math. Soc.*, vol. 3 (1902), pp. 285–301; *Bull. Amer. Math. Soc.*, vol. 9 (1902), pp. 394–401.

supposed. This may be reduced to the mere definition of a group by omitting §6 and §12 (the example being solved otherwise in §15), without breaking the continuity of the paper; but the reader acquainted with the rudiments of group-theory will find in the concrete nature of those sections excellent discipline for the later general theory, as well as an important theorem not elsewhere proved. The paper should prove of special interest to students of the theory of determinants.

In the second edition of his *Algebra* (pp. 193–218), Weber gives an account of the simplest results of the theory, following more closely the first method of Frobenius. Weber defines a character as a system of numbers satisfying a rather complicated system of equations, (51) below. The present paper, following more closely the second method of Frobenius, introduces the characters from the standpoint of their application to group-determinants, the definitions being explicit and very simple.

Properties of the Group-Matrix

1. To the h elements E (the identity), A, B, C, . . . of a finite group H, we make correspond h independent variables $x_E, x_A, x_B, x_C, \ldots$, in such a way that $x_L = x_{MN}$ if $L = MN$ in the group. By the *group-matrix* is meant

$$(1) \qquad \begin{pmatrix} x_{EE^{-1}} & x_{EA^{-1}} & x_{EB^{-1}} & \ldots & x_{EQ^{-1}} & \ldots \\ x_{AE^{-1}} & x_{AA^{-1}} & x_{AB^{-1}} & \ldots & x_{AQ^{-1}} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{PE^{-1}} & x_{PA^{-1}} & x_{PB^{-1}} & \ldots & x_{PQ^{-1}} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \equiv (x_{PQ^{-1}}) \equiv (x).$$

The table of subscripts is the body of a left-hand multiplication table for the group H. Thus, for a cyclic group of order 3 composed of the elements E, A, B ($A^3 = E$, $B^2 = A$), the multiplication table and the corresponding group-matrix are, respectively,

	$E^{-1} = E$	$A^{-1} = B$	$B^{-1} = A$
E	$EE^{-1} = E$	$EA^{-1} = B$	$EB^{-1} = A$
A	$AE^{-1} = A$	$AA^{-1} = E$	$AB^{-1} = B$
B	$BE^{-1} = B$	$BA^{-1} = A$	$BB^{-1} = E$

$$\begin{pmatrix} x_E & x_B & x_A \\ x_A & x_E & x_B \\ x_B & x_A & x_E \end{pmatrix}.$$

The determinant $\Theta(x) \equiv |x_{PQ^{-1}}|$ of matrix (1) is called the *group-deter-*

minant of H. Let $\Theta_{P,Q}$ be the adjoint (first minor with proper sign) of the element $x_{PQ^{-1}}$ in the group-determinant; the minor is obtained by erasing the row defined by P and the column defined by Q^{-1}.

THEOREM. *The value of $\Theta_{P,Q}$ depends only upon the product $PQ^{-1} \equiv C$ and not upon P and Q separately.*

After the replacement of the products giving the subscripts in (1) by the equivalent single elements, the variable x_C occurs once and but once in each row (or column) of the matrix; for the h positions of x_C, the adjoint will be shown to have the same value. Consider first the adjoint $\Theta_{P,P}$ of the variable $x_{PP^{-1}}$ in the main diagonal:

$$\Theta_{P,P} \equiv |x_{RS^{-1}}| \qquad (R, S = E, A, B, \ldots, \text{with } P \text{ excluded}).$$

Set $R = UP$, $S = VP$, so that $RS^{-1} = UPP^{-1}V^{-1} = UV^{-1}$. Then

$$\Theta_{P,P} = |x_{UV^{-1}}| \equiv \Theta_{E,E} \qquad (U, V = A, B, \ldots, \text{with } E \text{ excluded}).$$

To show next that $\Theta_{P,Q} = \Theta_{C,E}$ if $PQ^{-1} = C$, we set

$$x_{CRS^{-1}} = x'_{RS^{-1}} \qquad (R, S = E, A, B, \ldots).$$

Since CR and R run simultaneously through the series $E, A, B, \ldots$ of the h elements of the group H,

$$\pm |x_{RS^{-1}}| = |x_{CRS^{-1}}| \equiv |x'_{RS^{-1}}| \qquad (R, S = E, A, B, \ldots).$$

To obtain the term $x_{PQ^{-1}} \equiv x_C$ in the determinant $|x_{CRS^{-1}}|$, we must take $R = S$, so that the corresponding term in the determinant $|x'_{RS^{-1}}|$ is in the main diagonal. But the adjoints of the latter terms have been shown to be equal. Hence the same is true for the former terms, so that $\Theta_{P,Q} = \Theta_{C,E}$.

In view of the preceding theorem, we may introduce the notation $\Theta_{PQ^{-1}}$ for $\Theta_{P,Q}$. The result may therefore be stated in the form

$$\Theta_{PQ^{-1}} = \frac{1}{h} \frac{\partial \Theta(x)}{\partial x_{PQ^{-1}}}. \tag{2}$$

DEFINITION. The matrix $(\Theta_{PQ^{-1}})$ is the *adjoint matrix* of the group-matrix $(x_{PQ^{-1}})$.

EXAMPLE. For the above cyclic group $\{E, A, B\}$, the adjoint matrix is

$$\begin{pmatrix} \Theta_E & \Theta_B & \Theta_A \\ \Theta_A & \Theta_E & \Theta_B \\ \Theta_B & \Theta_A & \Theta_E \end{pmatrix} \equiv \begin{pmatrix} x_E^2 - x_A x_B & x_B^2 - x_E x_A & x_A^2 - x_E x_B \\ x_A^2 - x_E x_B & x_E^2 - x_A x_B & x_B^2 - x_E x_A \\ x_B^2 - x_E x_A & x_A^2 - x_E x_B & x_E^2 - x_A x_B \end{pmatrix}.$$

Formula (2) is now evident since $\Theta(x) \equiv x_E^3 + x_A^3 + x_B^3 - 3x_E x_A x_B$.

DEFINITION. By the *symmetry* of the group-matrix of a given group H is meant the property that from the order of the variables in its first row follows definitely the order of the variables in each of its rows.

The above theorem may now be stated in the following form:

The group-matrix and its adjoint matrix possess the same symmetry.

Thus, for the cyclic group of order 3, the elements of both the group-matrix and the adjoint matrix have the subscripts E, B, A in the first row; A, E, B in the second row; B, A, E in the third row.

2. Let $y_E, y_A, y_B, \ldots$ be a second set of h independent variables and consider the matrix $(y) \equiv (y_{PQ^{-1}})$ with the same table of subscripts as matrix (1). From the two systems of variables x_R, y_R we build a third system z_R as follows:

$$z_T = \sum_R x_R y_S \equiv \sum_S x_R y_S \qquad (RS = T). \tag{3}$$

In the first sum, R runs through the series $E, A, B, \ldots$ of the h elements of H, and for each R the element $R^{-1}T$ is to be taken for S. Similarly for the second sum. By the composition of the two matrices, we find that

$$(x)(y) = (z) \equiv (z_{P,Q}), \tag{4}$$

$$z_{P,Q} \equiv \sum_R x_{PR^{-1}} y_{RQ^{-1}} \qquad (R = E, A, B, \ldots).$$

Set $RQ^{-1} = S$. Then R and S run simultaneously through the series $E, A, B, \ldots$ of the h elements of H. Applying also the definition (3), we get

$$z_{P,Q} = \sum_S x_{PQ^{-1}S^{-1}} y_S = z_{PQ^{-1}} \qquad (S = E, A, B, \ldots).$$

Hence the matrix (z) *possesses the same symmetry as the matrices* $(x), (y)$.

3. As to the content of the preceding theorems a suggestive illustration is afforded by the following property, not mentioned by Frobenius and not essential in his theory, but forming the foundation of the investigations of Burnside and the writer. Consider the linear homogeneous transformation X whose coefficients are the elements $x_E, x_A, \ldots$ of the matrix (x). By allowing the variables $x_E, x_A, \ldots$ to take in turn every set of values in a given field (domain of rationality) F such that the determinant $\Theta(x)$ is not zero, we obtain a set of transformations which form a *group*. Indeed, by the preceding theorems, the inverse of X belongs to the set and the compound YX

of the transformations Y, X having the respective matrices (y), (x) is a transformation Z having the matrix (z) and consequently belonging to the set.

In view of the fact that the matrices (1) form a group, whereas other matrices of similar type do not, the group-matrix (1) is evidently of special importance in the study of the group H.

The Irreducible Factors of the Group-Determinant

4. Applying to the determinants of the matrices (4) the multiplication theorem, we get

$$\Theta(z) = \Theta(x)\,\Theta(y), \qquad \text{if } (z) = (x)(y). \tag{5}$$

Let the homogeneous integral rational function $\Theta(x)$ be decomposed into irreducible factors, each a homogeneous integral rational function, possibly with irrational coefficients:

$$\Theta(x) = \Phi^{e}\Phi'^{e'}\Phi''^{e''}\cdots, \qquad \Phi \equiv \Phi(x_E, x_A, x_B, \cdots), \tag{6}$$

and let $f, f', f'', \ldots$ be the degrees of $\Phi, \Phi', \Phi'', \ldots$, respectively. The diagonal terms and no others in $\Theta(x)$ are x_E. Hence $\Theta(x)$ reduces to x_E^h if zero be substituted for each variable except x_E; whence $\Phi(x)$ reduces to cx_E^f, where $c \neq 0$. By choice of the undetermined constant factor of $\Phi(x)$, we can make $c = 1$. Among the factors $\Phi, \Phi', \ldots$, occurs the linear factor $x_E + x_A + x_B + \cdots$; indeed, this sum occurs throughout the first row of a determinant derived from $\Theta(x)$ upon adding to the first row of the latter all the remaining rows.

5. Theorem. *The irreducible factors of the group-determinant* $\Theta(x)$ *possess the characteristic property*

$$\Phi(z) = \Phi(x)\Phi(y), \qquad \text{if } (z) = (x)(y). \tag{7}$$

If $\Phi(z)$ is an irreducible factor of $\Theta(z)$, it must, in view of (5), equal the product of a function $\Phi_1(x)$ of $x_E, x_A, \ldots$ alone and a function $\Phi_2(y)$ of $y_E, y_A, \ldots$ alone. Taking $x_E = 1$, $x_A = x_B = \cdots = 0$, we get $(z) = (y)$, whence $\Phi(y) = \Phi_1(1, 0, 0, \cdots)\Phi_2(y)$. Taking next $y_E = 1$, $y_A = y_B = \cdots = 0$, we get $(z) = (x)$, whence $\Phi(x) = \Phi_1(x)\Phi_2(1, 0, 0, \ldots)$. By choice of the undetermined constant factor of $\Phi_1(x)$, we can make $\Phi_1(1, 0, 0, \cdots) = 1$. Since $c \equiv \Phi(1, 0, 0, \cdots) = 1$, we have $\Phi_2(1, 0, 0, \cdots) = 1$. Hence $\Phi_1(x) = \Phi(x)$, $\Phi_2(y) = \Phi(y)$, so that (7) follows.

Inversely, let $\Phi(x_E, x_A, \ldots)$ be an irreducible homogeneous integral

rational function which satisfies relation (7). Taking for (y) the adjoint matrix to (x), we get $z_E = \Theta(x)$, $z_A = z_B = \cdots = 0$, whence $\Phi(z) = [\Theta(x)]^f$, and $\Phi(y)$ becomes an integral rational function of $x_E, x_A, \ldots$. Hence, by (7), the irreducible function $\Phi(x)$ is a factor of $\Theta(x)$.

The remarkable property (5) enjoyed by the group-determinant $\Theta(x)$ and the analogous property (7) enjoyed by an irreducible factor of $\Theta(x)$ play a fundamental rôle in the present theory. These properties emphasize the special significance of the function called a group-determinant among the totality of integral functions of the variables $x_E, x_A, \ldots$

To effect the complete decomposition of the group-determinant $\Theta(x)$, it is found convenient to fix the attention on certain coefficients in the (as yet unknown) irreducible factors of $\Theta(x)$ and call them *group-characters*. This is done for the linear factors in §6 and for general factors in §7. In terms of them all the remaining coefficients of the desired factors will be found to be readily expressible.

6. With the aid of (7), we can determine all the linear factors

$$\Phi(x) \equiv \sum_R \chi(R)\, x_R \qquad (R = E, A, B, \cdots), \tag{8}$$

where the h coefficients are considered to be values of a function χ, the value of $\chi(E)$ being unity. By (7) in connection with (3), we have

$$(\sum_R \chi(R)x_R)(\sum_S \chi(S)y_S) = \sum_T \chi(T)z_T = \sum_{R,S} \chi(RS)x_R y_S.$$

Equating the coefficients of $x_R y_S$ in the first and third sums, we get

$$\chi(RS) = \chi(R)\,\chi(S). \tag{9}$$

Thus $\chi(R^2) = [\chi(R)]^2$ and by induction $\chi(R^r) = [\chi(R)]^r$. Hence, if R is of period r, the left member is unity, so that $\chi(R)$ is an rth root of unity. Also $\chi(R)\,\chi(R^{-1}) = \chi(E) = 1$, so that

$$\chi(R^{-1}S^{-1}RS) = \chi(R^{-1})\,\chi(S^{-1})\,\chi(R)\,\chi(S) = 1.$$

Hence $\chi(F) = 1$ for every *commutator* $F = R^{-1}S^{-1}RS$ of the group H. Since

$$T^{-1}FT = (T^{-1}RT)^{-1}(T^{-1}ST)^{-1}(T^{-1}RT)(T^{-1}ST),$$

every conjugate within H of a commutator is itself a commutator. Hence the group G of order g generated by all the commutators of H is an invariant

subgroup of H. Let $E, A, B, C, \ldots$ be the right-hand multipliers which give rise to a rectangular array for H with the elements of G in the first row. Upon the h/g rows $GE, GA, GB, \ldots$ as elements, we have a *group*, since $(GA)(GB) = GAB$, in view of the relation $AGA^{-1} = G$. This group is called the *quotient-group* H/G. It is here a *commutative* group. For, if GA and GB are any two of its elements, there exists in the commutator group G the element $F_1 = BAB^{-1}A^{-1}$, for which

$$GBA = GF_1AB = GAB, \qquad (GB)(GA) = (GA)(GB).$$

Since $\chi(F) = 1$, if F is any element of G,

$$\chi(FA) = \chi(F)\,\chi(A) = \chi(A).$$

Hence $\chi(R)$ has the same value for all the elements R of the row GA. Hence there are h/g such values $\chi(E), \chi(A), \chi(B), \ldots$, defined by the respective rows $GE, GA, GB, \ldots$. Moreover, for F and F' chosen arbitrarily from G, $\chi(FA \cdot F'B)$ has a constant value $\chi(A)\,\chi(B)$. Hence there is defined a function χ whose h/g values for the quotient-group H/G have the property (9). Such a function χ is called a *character* of this commutative group H/G.

Consider, in general, a *commutative* group N of order n. By an elementary, but fundamental, theorem* on commutative groups, a system of elements $A_1, A_2, \ldots, A_\nu$ of periods $a_1, a_2, \ldots, a_\nu$, respectively, can be chosen such that every element A of N can be expressed in one and only one way in the form

$$A = A_1^{\alpha_1}A_2^{\alpha_2} \cdots A_\nu^{\alpha_\nu} \qquad (0 \leqq \alpha_1 < a_1, \cdots, 0 \leqq \alpha_\nu < a_\nu).$$

That these elements A form a commutative group of order $n = a_1a_2 \cdots a_\nu$, in case $A_1, A_2, \ldots, A_\nu$ are independent commutative elements, is directly evident. To the elements $R, S, \ldots$ of the group N we make correspond any numbers $\chi(R), \chi(S), \ldots$ such that property (9) holds and such that $\chi(R)$ is zero for no element R. Setting $S = E$, the identity, we get, from (9), $\chi(R) = \chi(R)\,\chi(E)$, whence $\chi(E) = 1$. It follows, as at the beginning of the section, that the number $\chi(R)$ is an rth root of unity, r being the period of the element R. The function χ is called a *character* of the commutative group N. Two characters χ and χ_1 are said to be different when $\chi(A) \neq \chi_1(A)$

* Weber, *Algebra*, II, Zweiter Abschnitt; Burnside, *The Theory of Groups*, p. 50.

for at least one element A of N. It is next shown that *the number of different characters equals the order n of the group N*. Set $\chi(A_1) = w_1, \cdots, \chi(A_\nu) = w_\nu$. Then by (9)

$$\chi(A) \equiv \chi(A_1^{a_1} A_2^{a_2} \cdots A_\nu^{a_\nu}) = w_1^{a_1} w_2^{a_2} \cdots w_\nu^{a_\nu}. \tag{10}$$

But $w_1^{a_1} = 1, \cdots, w_\nu^{a_\nu} = 1$, a_1 being the period of $A_1, \cdots, a_\nu$ the period of A_ν. Hence there are at most $n = a_1 a_2 \cdots a_\nu$ different characters. To show that there are at least n different characters, consider an arbitrary root w_i of $w_i^{a_i} = 1$ for $i = 1, \cdots, \nu$, and define $\chi(A)$ by formula (10). This function χ evidently satisfies the relation (9), while $\chi(A)$ is zero for no element A in the group. Hence χ defines a character. Moreover, a second set of roots $w_1', \ldots, w_\nu'$ leads to a different character χ', since

$$w_1'^{\alpha_1} \cdots w_\nu'^{\alpha_\nu} \equiv w_1^{\alpha_1} \cdots w_\nu^{\alpha_\nu} \qquad \text{(for every } \alpha_1, \cdots, \alpha_\nu)$$

requires $w_1' = w_1$ (by taking $\alpha_1 = 1, \alpha_2 = 0, \cdots, \alpha_\nu = 0), \cdots, w_\nu' = w_\nu$.

Returning to the commutative group H/G of order h/g, let ψ be one of its h/g characters and set $\chi(R) = \psi(GA)$ for every element R of the row GA of the rectangular array. Then the function (8), with the value just assigned to $\chi(R)$, is a factor of the group-determinant $\Theta(x)$. Indeed, if we set $y_R = \chi(R) x_R$, we find that

$$y_{PQ^{-1}} = \chi(PQ^{-1}) x_{PQ^{-1}} = \chi(P)\chi(Q^{-1}) x_{PQ^{-1}}, \qquad |y_{PQ^{-1}}| = |x_{PQ^{-1}}|.$$

But the first determinant was shown to have the factor Σy_R by adding to its first row the remaining rows. After the removal of this factor, the elements of the first row all equal 1. By subtracting the first column from the remaining columns, we see that the resulting determinant depends only upon the differences $y_R - y_S$ and hence is not divisible by the sum Σy_R. Hence the second determinant $|x|$ has the factor $\Sigma \chi(R) x_R$ and contains it only to the first power. We have, therefore, proved the following

THEOREM. *The linear factors of the group-determinant enter to the first power only. Their number is the quotient of the order of H by the order of its commutator group G.*

EXAMPLE. If H is a cyclic group, it is commutative and the order of G is unity. Then $\Theta(x)$ is a so-called cyclic determinant of order h and has h distinct linear factors.

For the cyclic group H_3, the factors and the corresponding characters are

	$\chi(E)$	$\chi(A)$	$\chi(B)$
$x_E + x_A + x_B$,	1	1	1
$x_E + \rho x_A + \rho^2 x_B$,	1	ρ	ρ^2
$x_E + \rho^2 x_A + \rho x_B$,	1	ρ^2	ρ

where $\rho \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ is an imaginary cube root of unity.

7. Let Φ be an integral homogeneous function of degree f in the variables $x_E, x_A, x_B, \ldots$, which satisfies condition (7). In it the coefficient of x_E^f is unity. In proof, we take $y_E = 1, y_A = y_B = \cdots = 0$, whence $(z) = (x)$, $\Phi(x) = \Phi(x)\,\Phi(1, 0, 0, \ldots)$. Hence $\Phi(1, 0, 0, \ldots)$, the coefficient of x_E^f in $\Phi(x)$, is unity. Denote by $\chi(R)$ the coefficient of x_E^{f-1} in $\dfrac{\partial \Phi}{\partial x_R}$, so that, for $R \neq E$, $\chi(R)$ is the coefficient of $x_E^{f-1} x_R$ in Φ, while

$$\chi(E) = f. \tag{11}$$

The h constants $\chi(R)$ are considered to be values of a function χ. If Φ is irreducible, χ is called the *character of degree f* which corresponds to the factor Φ. As shown in §6, the characters of degree $f = 1$ are roots of unity.

Consider the following expansion, u being a variable,

$$\Phi(x_E + u, x_A, x_B, \cdots) = u^f + \Phi_1 u^{f-1} + \Phi_2 u^{f-2} + \cdots + \Phi_f, \tag{12}$$

where Φ_m is an integral homogeneous function of degree m of $x_E, x_A, x_B, \ldots$. In particular, $\Phi_f = \Phi$ and

$$\Phi_1 = \sum_R \chi(R)\, x_R \qquad (R = E, A, B, \ldots). \tag{13}$$

If the right member of (12) vanishes for $u = -u_1, -u_2, \cdots, -u_f$, then

$$\Phi(x_E + u, x_A, x_B, \cdots) \equiv (u + u_1)(u + u_2) \cdots (u + u_f). \tag{14}$$

In an analogous manner, we set

$$u^n + v \equiv (u + v_1)(u + v_2) \cdots (u + v_n). \tag{15}$$

If the group-matrix (x) be compounded with itself n times, there results, as in §2, a matrix $(x)^n = (x_{PQ^{-1}}^{(n)}) \equiv (x^{(n)})$, where

$$x_R^{(n)} = \sum_{R_1, \ldots, R_n} x_{R_1} x_{R_2} \cdots x_{R_n} \qquad (R_1 R_2 \cdots R_n = R). \tag{16}$$

Let (ϵ) denote the unit-matrix, whose elements are all zero except those in the main diagonal which are unity. Then from (15) follows*

$$(x)^n + v(\epsilon) \equiv (x^{(n)}) + v(\epsilon) \equiv [(x) + v_1(\epsilon)] \cdot \cdots \cdot [(x) + v_n(\epsilon)].$$

Since the matrices $[(x) + v_i(\epsilon)]$ possess the same symmetry as the group matrix (x) we may apply to the former the relation (7). Hence

$$\Phi(x_E^{(n)} + v,\ x_A^{(n)}, x_B^{(n)}, \cdots) = \prod_{i=1}^{n} \Phi(x_E + v_i, x_A, x_B, \cdots)$$

$$= \prod_{i=1}^{n}(v_i + u_1)(v_i + u_2) \cdots (v_i + u_f) = (v + u_1^n)(v + u_2^n) \cdots (v + u_f^n),$$

upon applying (14) and (15). The coefficient of v^{f-1} in the last product is

$$u_1^n + u_2^n + \cdots + u_f^n \equiv S_n.$$

In view of (12) and (13), the coefficient of v^{f-1} in $\Phi(x_E^{(n)} + v, x_A^{(n)}, \ldots)$ is $\Sigma\chi(R)x_R^{(n)}$. Hence

$$S_n = \sum_R \chi(R) x_R^{(n)} \qquad (R = E, A, B, \cdots). \tag{17}$$

In view of (16), this result may be written

$$S_n = \sum_{R_1, \ldots, R_n} \chi(R_1 R_2 \cdots R_n) x_{R_1} x_{R_2} \cdots x_{R_n} \quad (R_i = E, A, B, \cdots). \tag{18}$$

The coefficients $\Phi_1, \Phi_2, \ldots$ of (12), in particular the desired function $\Phi_f \equiv \Phi$, can be expressed in terms of the sums of powers $S_1, S_2, \ldots, S_n, \ldots$ of the roots with their signs changed (compare Serret, *Algèbre supérieure*, 4th ed., vol. 1, p. 460). For example

$$\begin{aligned} 2\Phi_2 &= S_1^2 - S_2, \quad 6\Phi_3 = S_1^3 - 3S_1S_2 + 2S_3, \\ 24\Phi_4 &= S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4. \end{aligned} \tag{19}$$

Properties of Group-Characters.

8. For $y_A = 1$ and the remaining $y_S = 0$, (3) gives $z_T = x_{TA^{-1}}$, and $\Phi(y)$ reduces to a constant $\psi(A)$. Then relation (7) becomes

$$\Phi(x_{TA^{-1}}) = \psi(A)\Phi(x_T). \tag{20}$$

For $x_A = 1$, $x_R = 0$ $(R \neq A)$, (3) gives $z_T = y_{A^{-1}T}$. Then (7) becomes

$$\Phi(y_{A^{-1}T}) = \psi(A)\Phi(y_T). \tag{21}$$

* By a principle of the theory of matrices, an identity between two integral rational functions of a variable u leads to an analogous identity obtained by replacing u by a matrix (x) and the constant term $C \equiv Cu^0$ by $C(x)^0 \equiv C(\epsilon)$.

Replacing T by AT in (21), and then A by A^{-1}, we get

$$\Phi(y_T) = \psi(A)\Phi(y_{AT}) = \psi(A)\ \psi(A^{-1})\ \Phi(y_T).$$

Hence $\psi(A)\ \psi(A^{-1}) = 1$. Replacing y_T by x_{TA} in (21), and using (20) we get

$$\Phi(x_{A^{-1}TA}) = \psi(A)\ \Phi(x_{TA}) = \psi(A)\ \psi(A^{-1})\ \Phi(x_T),$$

$$\Phi(x_{A^{-1}TA}) = \Phi(x_T). \tag{22}$$

Equating the coefficients of $x_E^{-1}x_R$, we get $\chi(ARA^{-1}) = \chi(R)$. *Hence a character has the same value for all elements conjugate within* H. Setting $R = BA$, we get

$$\chi(AB) = \chi(BA) \qquad (A, B \text{ any elements of } H). \tag{23}$$

From the usual relations between the elements of a determinant Θ and its adjoint minors, we have

$$\sum_R x_{PR^{-1}}\Theta_{QR^{-1}} = \sum_R x_{RP^{-1}}\Theta_{RQ^{-1}} \equiv \begin{cases} \Theta \text{ if } P = Q, \\ 0 \text{ if } P \neq Q. \end{cases}$$

Setting $T = QR^{-1}$, $A = PQ^{-1}$ in the first sum, but $T = RQ^{-1}$, $A = QP^{-1}$ in the second sum, and applying formula (2), we get

$$\sum_T x_{AT}\frac{\partial\Theta}{\partial x_T} = \sum_T x_{TA}\frac{\partial\Theta}{\partial x_T} = \epsilon_A h\Theta, \tag{24}$$

where $\epsilon_A = 1$ if $A = E$, the identity, while $\epsilon_A = 0$ if $A \neq E$. In view of (6), $\Theta = \Phi^e\Psi$, where Φ and Ψ have no common factor. Hence

$$\sum_T x_{AT}\left(e\Psi\frac{\partial\Phi}{\partial x_T} + \Phi\frac{\partial\Psi}{\partial x_T}\right) = \epsilon_A h\Phi\Psi.$$

Hence $\Sigma x_{AT}\dfrac{\partial\Phi}{\partial x_T}$ is divisible by Φ, and the quotient is a constant, being of degree zero. The latter is $\chi(A^{-1})$, as shown by comparing the coefficients of x'_E. Hence*

$$\sum_T x_{AT}\frac{\partial\Phi}{\partial x_T} = \sum_T x_{TA}\frac{\partial\Phi}{\partial x_T} = \chi(A^{-1})\Phi. \tag{25}$$

* Another proof is given by differentiating the terms of (7) with respect to y_A and setting $y_R = \epsilon_R$ in the result (and with respect to x_A and setting $x_R = \epsilon_R$), applying (3) in each case.

Multiplying the first and third members by $h\Theta_{AS}$ and summing with respect to A,

$$h\sum_T\Big(\sum_A x_{AT}\Theta_{AS}\Big)\frac{\partial\Phi}{\partial x_T}\equiv h\Theta\,\frac{\partial\Phi}{\partial x_S}=\Phi\sum_A\chi(A^{-1})\,\frac{\partial\Theta}{\partial x_{AS}}.$$

Setting first $\Theta=\Phi^e\Psi$ and replacing AS by R in the last member, we get

$$h\Psi\frac{\partial\Phi}{\partial x_S}=\sum_R\chi(SR^{-1})\left(e\Psi\frac{\partial\Phi}{\partial x_R}+\Phi\frac{\partial\Psi}{\partial x_R}\right).$$

The terms which multiply Ψ are equal, since their difference is divisible by Φ, while their degree is $f-1$. Hence

$$\frac{h}{e}\frac{\partial\Phi}{\partial x_S}=\sum_R\chi(SR^{-1})\,\frac{\partial\Phi}{\partial x_R}. \tag{26}$$

Setting next $\Theta=\Phi'^{e'}\Psi'$, Φ' being an irreducible factor distinct from Φ, we get

$$h\Phi'\Psi'\frac{\partial\Phi}{\partial x_S}=\Phi\sum_R\chi(SR^{-1})\left(e'\Psi'\frac{\partial\Phi'}{\partial x_R}+\Phi'\frac{\partial\Psi'}{\partial x_R}\right).$$

Hence the following sum is divisible by Φ' and therefore vanishes:

$$\sum_R\chi(SR^{-1})\frac{\partial\Phi'}{\partial x_R}=0. \tag{27}$$

Comparing the coefficients of x_E^{f-1} in (26) and those of $x_E^{f'-1}$ in (27), we get

$$\sum_R\chi(SR^{-1})\chi(R)=\frac{h}{e}\,\chi(S),\qquad \sum_R\chi(SR^{-1})\psi(R)=0, \tag{28}$$

where $\psi(R)$ is the character corresponding to the factor Φ'. For $S=E$,

$$\sum_R\chi(R)\chi(R^{-1})=\frac{hf}{e},\qquad \sum_R\psi(R)\chi(R^{-1})=0. \tag{29}$$

Hence the values $\chi(E),\chi(A),\ldots$ are not proportional to $\psi(E),\psi(A),\ldots$. Replacing R by RB and S by AB in (28),

$$\sum_R\chi(AR^{-1})\,\chi(BR)=\frac{h}{e}\,\chi(AB). \tag{28'}$$

Another important property of characters is derived in §10.

9. The *special group-matrix* (η) is obtained from (y) by setting

$$\eta_R\equiv y_R=y_{A^{-1}RA}=y_{B^{-1}RB}=y_{C^{-1}RC}=\cdots, \tag{30}$$

so that the number of distinct elements η_R equals the number r of sets of conjugate operators of the group H. The determinant $|\eta|$ is called the *special group-determinant.** Then (η) is commutative with any matrix (x) of the form (1). Indeed, by §2 $(x)(y)$ is of the form (z), where

$$z_{PQ^{-1}} = \sum_S x_{PQ^{-1}S^{-1}} y_S \equiv \sum_S x_S y_{S^{-1}PQ^{-1}},$$

upon replacing S by $S^{-1}PQ^{-1}$ in the second sum. Also $(y)(x)$ is of the form (z'), where

$$z'_{PQ^{-1}} = \sum_S y_{PQ^{-1}S^{-1}} x_S = \sum_S y_{S^{-1}(PQ^{-1}S^{-1})S} x_S = z_{PQ^{-1}}.$$

Hence an arbitrary group-matrix $(x_{PQ^{-1}})$, a special group-matrix $(y_{PQ^{-1}})$, and $(\epsilon_{PQ^{-1}})$, the unit-matrix, are commutative with each other. By a theorem of Frobenius,† the determinant

$$|ux_{PQ^{-1}} + vy_{PQ^{-1}} + w\epsilon_{PQ^{-1}}|$$

is a product of linear homogeneous functions of the variables u, v, w. Hence

$$(31)\quad \Phi(ux_E + vy_E + w,\ ux_A + vy_A,\ ux_B + vy_B,\ \cdots) = \prod_{i=1}^{f} (u_i u + v_i v + w),$$

the coefficients of w being chosen to be unity in view of the term w^f on the left. Setting $u = 1$, $v = 0$, we obtain (compare (14) of §7)

$$(32)\quad \Phi(x_E + w,\ x_A,\ x_B,\ \cdots) = (u_1 + w)(u_2 + w)\cdots(u_f + w),$$

so that $u_1, u_2, \ldots, u_f$ depend only upon the variables $x_E, x_A, \ldots$ Similarly, $v_1, v_2, \ldots, v_f$ depend only upon $y_E, y_A, \ldots$ Since $\Phi(x)$ is irreducible, the function (32) is irreducible, considered as a function of w, *i.e.*, it is not expressible as a product of two integral rational functions of w whose coefficients are rational functions of $x_E, x_A, \ldots$. Replacing w by $w + v_1$ in (32), we obtain the irreducible function of w,

$$\Phi(x_E + v_1 + w, x_A, x_B, \cdots) = (u_1 + v_1 + w)(u_2 + v_1 + w)\cdots(u_f + v_1 + w).$$

It has the factor $u_1 + v_1 + w$ in common with $\Phi(x_E + y_E + w,\ x_A + y_A, \cdots)$, obtained by setting $u = v = 1$ in (31). Hence the two functions are identical. Taking $w = 0$, we get

$$(33)\quad \Phi(x_E + y_E,\ x_A + y_A,\ x_B + y_B,\ \cdots) = \Phi(x_E + v_1,\ x_A,\ x_B,\ \cdots),$$

---

* Inversely, a matrix (y) with elements in a field F and commutative with every matrix (x) in F is a special matrix (η), *Transactions*, *l. c.*, §2.

† Über vertanschbare Matrizen, *Berliner Sitzungsberichte*, 1896, p. 602.

v_1 being a function of $y_E, y_A, \cdots$ Setting $x_E = u$, $x_A = x_B = \cdots = 0$, we get

$$\Phi(y_E + u, y_A, y_B, \cdots) = (u + v_1)^f. \tag{34}$$

Equating the coefficients of u^{f-1} and applying the definitions of §7, we get

$$\sum_R \chi(R) y_R = f v_1. \tag{35}$$

An irreducible factor Φ of degree f of the general group-determinant becomes, upon imposing the conditions (30), the fth power of a linear factor $\frac{1}{f}\sum\chi(R)y_R$ of the special group-determinant.

Since the values of the character $\psi(R)$ corresponding to an irreducible factor Φ' distinct from Φ are not proportional to the values of $\chi(R)$ corresponding to Φ (end of §8), the linear function $\frac{1}{f'}\sum\psi(R)y_R$ is not a constant times the linear function $\frac{1}{f}\sum\chi(R)y_R$.

Two distinct irreducible factors of the general group-determinant give rise to two independent linear factors of the special group-determinant.

10. Let the elements $E, A, B, \ldots$ of the group H be separated into distinct sets of conjugates and designate the sets as the 1st, 2d, $\ldots$, ρth, $\ldots$, rth set. Let h_ρ denote the number of elements R in the ρth set; for each of them $\chi(R)$ has the same value χ_ρ. In view of (30), there are r distinct variables $y_1, y_2, \ldots, y_r$. Then (35) becomes

$$\sum_{\rho=1}^{r} h_\rho \chi_\rho y_\rho = f v_1. \tag{35'}$$

Differentiating (33) with respect to y_ρ and setting $y_1 = \cdots = y_r = 0$ in the result, we get

$$\sum_R^{\rho\text{th set}} \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_R} = \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_E}\frac{\partial v_1}{\partial y_\rho} = \frac{1}{f} h_\rho \chi_\rho \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_E}.$$

Let A be a fixed element and replace each x_R by x_{AR}. By (21), $\Phi(x_T)$ is thereby multiplied by a constant $\psi(A^{-1})$. It follows that

$$\sum_R^{\rho\text{th set}} \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_{AR}} = \frac{1}{f} h_\rho \chi_\rho \frac{\partial\Phi(x_E, x_A, \ldots)}{\partial x_A}.$$

Equating the coefficients of x_E^{f-1}, and replacing χ_ρ by $\chi(B)$, we get

$$(36) \qquad \sum_R^{\rho\text{th set}} \chi(AR) = \frac{1}{f} h_B \chi(B)\chi(A),$$

where R runs through h_B elements conjugate with B. Set $R = S^{-1}BS$, and let S run through the series of h elements of H, so that R becomes identical h/h_B times with each element conjugate with B. Hence the left member of (36) equals $\frac{1}{h} h_B \sum_S \chi(AS^{-1}BS)$, so that

$$(37) \qquad h\chi(A)\chi(B) = f\sum_R \chi(AR^{-1}BR).$$

11. Let l denote the number of distinct irreducible factors $\Phi, \Phi', \ldots$ of Θ. Corresponding to $\Phi^{(\lambda)}$, which occurs to the power $e^{(\lambda)}$ in Θ, there exists a character $\chi^{(\lambda)}$, for which $\chi^{(\lambda)}(E) = f^{(\lambda)}$, the degree of $\Phi^{(\lambda)}$. Since Θ is of degree h,

$$(38) \qquad \sum_\lambda e^{(\lambda)}\chi^{(\lambda)}(E) = \sum_\lambda e^{(\lambda)} f^{(\lambda)} = h.$$

By the definition in §7, $\chi^{(\lambda)}(R)$ is the coefficient of $x^{f^{(\lambda)}-1}x_R$ in $\Phi^{(\lambda)}$, if $R \neq E$; while the coefficient of $x_E^{f^{(\lambda)}}$ is 1. Hence the coefficient of $x_E^{h-1}x_R$ in the product (6) is $\sum e^{(\lambda)}\chi^{(\lambda)}(R)$. But x_E occurs only in the main diagonal of $\Theta(x)$, so that the coefficient of $x_E^{h-1}x_R$ is zero if $R \neq E$. Hence

$$(39) \qquad \sum_\lambda e^{(\lambda)}\chi^{(\lambda)}(R) = 0. \qquad (R \neq E).$$

By (38) and (39), we have

$$\sum_\lambda \frac{1}{h}\, e^{(\lambda)}\chi^{(\lambda)}(B^{-1}R^{-1}BR) = \begin{cases} 1 & \text{if } RB = BR, \\ 0 & \text{if } RB \neq BR. \end{cases}$$

Now the number of pairs of elements R, B of H for which $RB = BR$ is rh, where r is the number of sets of conjugates in H. Indeed, if R be a definite element of the ρth set there exist h/h_ρ elements B commutative with R. If R runs through all h_ρ elements of the ρth set, there results $h_\rho \cdot h/h_\rho$ pairs of solutions of $RB = BR$. This number h being the same for each of the r sets, there results hr pairs of solutions in all. Hence the sum (in which B, R run through the series of all elements of H)

$$(40) \qquad \sum_{B,R}\sum_\lambda \frac{1}{h}\, e^{(\lambda)}\chi^{(\lambda)}(B^{-1}R^{-1}BR)$$

equals hr. But, by (37) and (29), we have

$$\sum_B \left[\frac{e}{h} \sum_R \chi(B^{-1}R^{-1}BR)\right] = \frac{e}{f} \sum_B \chi(B^{-1})\chi(B) = h.$$

Since the first sum equals h for each of the l values of λ, the sum (40) equals hl. Hence $l = r$, which proves the following theorem:

The number of distinct irreducible factors of the group-determinant Θ *equals the number* r *of sets of conjugate elements in the group* H.

COROLLARY. The number of distinct linear factors of the special group-determinant equals the number r of sets of conjugate elements.

12. As an example, consider the symmetric group on 3 letters and set

$$E = \text{identity},\ A = (132),\ B = (123),\ C = (12),\ D = (13),\ F = (23).$$

The group-matrix (1) then becomes

$$(41) \qquad \begin{pmatrix} x_E & x_B & x_A & x_C & x_D & x_F \\ x_A & x_E & x_B & x_D & x_F & x_C \\ x_B & x_A & x_E & x_F & x_C & x_D \\ x_C & x_D & x_F & x_E & x_B & x_A \\ x_D & x_F & x_C & x_A & x_E & x_B \\ x_F & x_C & x_D & x_B & x_A & x_E \end{pmatrix}$$

The commutator group (§6) is an invariant subgroup and contains $C^{-1}B^{-1}CB = A$; hence it is the cyclic group $\{E, A, B\}$. Hence there are exactly 6/3 characters of the first degree. If χ be one of these two characters, we have

$$(42) \quad \chi(E) = \chi(A) = \chi(B) = 1. \qquad \chi(C) = \chi(D) = \chi(F) = \pm 1.$$

Hence the linear factors of the group-determinant Θ are

$$(43)\ x_E + x_B + x_A + x_C + x_D + x_F, \qquad x_E + x_B + x_A - x_C - x_D - x_F.$$

Since the number of sets of conjugate elements is $r = 3$, there is only one irreducible factor Φ of degree $f > 1$. It will be determined by a general method in §15. In the present simple case, it may be determined by inspection as follows. Let ω be an imaginary cube root of unity and set

$$x = x_E + \omega x_A + \omega^2 x_B,\ y = x_E + \omega^2 x_A + \omega x_B,\ z = x_C + \omega x_D + \omega^2 x_F,$$
$$w = x_C + \omega^2 x_D + \omega x_F.$$

Multiply the second row of Θ by ω and the third row by ω^2 and add the results to the first row; multiply the fifth row by ω^2 and the sixth row by ω

and add the results to the fourth row. The new first and fourth rows are

$$\begin{array}{cccccc} x & \omega x & \omega^2 x & z & \omega^2 z & \omega z, \\ w & \omega w & \omega^2 w & y & \omega^2 y & \omega y. \end{array}$$

In Laplace's development of the new determinant, each term is the product of a determinant of order two formed from the above two rows by the complementary determinant of order four. Since each of the former has the factor $xy - zw \equiv \Phi$, the determinant Θ has the factor Φ. Since x, y, z, w are independent functions of $x_E, x_A, \ldots, \Phi$ is irreducible. The value of Φ is

$$(44) \quad x_E^2 + x_A^2 + x_B^2 - x_E x_A - x_E x_B - x_A x_B - x_C^2 - x_D^2 - x_F^2 + x_C x_D + x_C x_F + x_D x_F.$$

Hence* Φ must occur in Θ to the power $e = 2$, so that $f = e = 2$. By §7, the corresponding character $\chi^{(2)}$ has the values

$$(45) \quad \chi^{(2)}(E) = 2, \quad \chi^{(2)}(A) = \chi^{(2)}(B) = -1, \quad \chi^{(2)}(C) = \chi^{(2)}(D) = \chi^{(2)}(F) = 0.$$

Equality of the Numbers e and f.

13. The most difficult part of the whole theory, as Frobenius states, is the proof that the degree f of an irreducible factor Φ of the group-determinant Θ equals the exponent e of the power to which Φ enters Θ.

For $f = 1$, the exponent e is 1 by §6.

For $f = 2$, relations (18) and (19) give

$$(46) \quad 2\Phi = S_1^2 - S_2 = \sum_{A,B} [\chi(A)\chi(B) - \chi(AB)] x_A x_B.$$

Now $\Phi_3, \Phi_4, \ldots$ do not occur in (12); but, if they be *defined* by (19), they are all zero. That $\Phi_3, \Phi_4, \ldots$ are identically zero, may be verified directly. Thus $6\Phi_3$ equals

$$S_1^3 - 3S_1S_2 + 2S_3 \equiv (u_1 + u_2)^3 - 3(u_1 + u_2)(u_1^2 + u_2^2) + 2(u_1^3 + u_2^3) \equiv 0.$$

In this we substitute the value of S_n given by (18) and determine the coefficient of $x_A x_B x_C$. If A, B, C are distinct there occurs a common factor

* In a note in the *American Mathematical Monthly*, March, 1902, I prove these results by the most elementary principles of determinants. They also follow by methods of group theory (Dedekind, *Berliner Sitzungsberichte*, 1897, p. 1007; Dickson, *Trans. Amer. Math. Soc.*, vol. 3 (1902), p. 296).

6; if two are equal, the factor 3; if all are equal, the factor 1. Cancelling the factor, we get

$$\chi(A)\,\chi(B)\,\chi(C)-\chi(A)\,\chi(BC)-\chi(B)\,\chi(AC)-\chi(C)\,\chi(AB)+\chi(ABC)+\chi(ACB)=0.$$

Replacing B by BC^{-1} and summing over all h elements C of H, we get

$$\chi(A)\sum_C \chi(BC^{-1})\,\chi(C)-h\chi(A)\,\chi(B)-\Sigma\,\chi(BC^{-1})\,\chi(AC)-\Sigma\,\chi(C)\,\chi(ABC^{-1})+h\chi(AB)+\Sigma\,\chi(ACBC^{-1})=0.$$

Applying relations (23), (28′), and (37), we get

$$\frac{h}{e}\chi(A)\,\chi(B)-h\chi(A)\,\chi(B)-\frac{h}{e}\chi(AB)-\frac{h}{e}\chi(AB)+h\chi(AB)+\frac{h}{f}\chi(A)\,\chi(B)=0.$$

Since $f=2$, this relation factors into

$$\left(\frac{h}{e}-\frac{h}{2}\right)\Big[\chi(A)\,\chi(B)-2\chi(AB)\Big]=0.$$

If the second factor vanished, we should obtain, upon multiplying it by x_Ax_B and summing over all pairs of elements A, B of the group, the result $S_1^2-2S_2=0$. Then would $4\Phi=S_1^2$, whereas Φ is irreducible.* Hence must $e=2=f$.

For $f=3$, we have by (19),

$$6\Phi=S_1^3-3S_1S_2+2S_3,\qquad 24\Phi_4=S_1^4-6S_1^2S_2+8S_1S_3+3S_2^2-6S_4\equiv 0. \tag{47}$$

For brevity set (R) for $\chi(R)$. The coefficient of $x_Ax_Bx_Cx_D$ in $24\Phi_4$ is (aside from a factor 24, 12, or 4):

$$\begin{aligned}&(A)(B)(C)(D)-(B)(C)(AD)-(A)(C)(BD)-(A)(B)(CD)\\&-(A)(D)(BC)-(B)(D)(AC)-(C)(D)(AB)+(A)(BCD)\\&+(B)(ACD)+(C)(ABD)+(D)(ABC)+(A)(BDC)+(B)(ADC)\\&+(C)(ADB)+(D)(ACB)+(BC)(AD)+(AC)(BD)+(AB)(CD)\\&-(ABCD)-(ACBD)-(BACD)-(BCAD)-(CABD)-(CBAD)\\&=0.\end{aligned}$$

* To give another proof, like that used for the case of $f=4$, we replace A by AB^{-1} and sum for B. We obtain $\sum_B[\chi(AB^{-1})\chi(B)-2\chi(A)]\equiv\chi(A)(h/e-2h)$, upon applying (28). But the last expression is not zero.

Replacing C by CD^{-1} and summing over all h elements D, and applying (23) (28′), (37), we get

$$\frac{h}{e}[(A)(B)(C)-2(B)(AC)-2(A)(BC)-(C)(AB)+3(ABC)+3(ACB)]+h[-(A)(B)(C)+(C)(AB)+(A)(BC)+B(AC)-(ABC)-(ACB)]+\frac{h}{f}[2(A)(B)(C)-(B)(AC)-(A)(BC)-2(C)(AB)]=0.$$

Since $f=3$, this relation factors into

$$\left(\frac{h}{e}-\frac{h}{3}\right)[(A)(B)(C)-2(A)(BC)-2(B)(AC)-(C)(AB)+3(ABC)+3(ACB)]=0.$$

If the second factor vanished identically, we should obtain, upon multiplying it by $x_Ax_Bx_C$ and summing over all triples A, B, C, the result $S_1^3-5S_1S_2+6S_3=0$, so that, by (47), $9\Phi=S_1(S_1^2-2S_2)$, contrary to the irreducibility of Φ. Hence $e=3=f$.

For $f=4$, we have $\Phi=\Phi_4$, defined by (47). Also

$$5!\,\Phi_5=S_1^5-10S_1^3S_2+20S_1^2S_3+15S_1S_2^2-30S_1S_4-20S_2S_3+24S_5\equiv 0.$$

For the coefficient of $x_Ax_Bx_Cx_Dx_F$ in $5!\Phi_5$, apart from a common numerical factor, we have 120 terms of which the first is $(A)(B)(C)(D)(F)$, while 10 are of the type $-(A)(B)(C)(DF)$, 20 of the type $(A)(B)(CDF)$, 15 of the type $(A)(BC)(DF)$, 30 of the type $-(A)(BCDF)$, 20 of the type $-(AB)(CDF)$, and 24 of the type $(ABCDF)$. Each term corresponds to a substitution on five letters, its sign being + or − according as the substitution is even or odd. We therefore take all permutations of the letters in any type, aside from cyclic interchanges within the cycles, which leave the character unaltered by (23). Replacing D by DF^{-1} and summing over all h elements F, and applying (23), (28′), (37), we get

$$\frac{h}{e}U+hV+\frac{h}{f}W=0, \tag{48}$$

$$\begin{aligned}U\equiv{}&(A)(B)(C)(D)-2(A)(C)(BD)-2(B)(C)(AD)\\&-2(A)(B)(CD)+(D)(BAC)+(D)(ABC)-(C)(D)(AB)\\&-(A)(D)(BC)-(B)(D)(AC)+2(AB)(CD)+2(AC)(BD)\\&+2(AD)(BC)+3(A)(BCD)+3(A)(CBD)+3(B)(ACD)\end{aligned}$$

$$+3(B)(CAD)+3(C)(ABD)+3(C)(BAD)-4(ABCD)$$
$$-4(ABDC)-4(ACBD)-4(ACDB)-4(ADBC)-4(ADCB).$$

$$V\equiv-(A)(B)(C)(D)+(A)(B)(CD)+(A)(C)(BD)+(B)(C)(AD)$$
$$+(C)(D)(AB)+(A)(D)(BC)+(B)(D)(AC)-(A)(BCD)$$
$$-(A)(CBD)-(B)(ACD)-(B)(ADC)-(C)(ABD)$$
$$-(C)(ADB)-(D)(ABC)-(D)(ACB)-(AB)(CD)$$
$$-(AC)(BD)-(AD)(BC)+(ABCD)+(ABDC)+(ACBD)$$
$$+(ACDB)+(ADBC)+(ADCB).$$

$$W\equiv3(A)(B)(C)(D)-2(A)(B)(CD)-2(A)(C)(BD)$$
$$-2(B)(C)(AD)-3(A)(D)(BC)-3(B)(D)(AC)-3(C)(D)(AB)$$
$$+2(AB)(CD)+2(AC)(BD)+2(AD)(BC)+3(D)(ABC)$$
$$+3(D)(ACB)+(A)(BCD)+(A)(CBD)+(B)(ACD)$$
$$+(B)(CAD)+(C)(ABD)+(C)(BAD).$$

Since $f=4$, the left member of (48) factors into $h/e-h/4$ and U. But if $U\equiv0$,

$$\sum_{A,B,C,D}Ux_Ax_Bx_Cx_D\equiv S_1^4-9S_1^2S_2+6S_2^2+20S_1S_3-24S_4=0.*$$

Then, Φ being the Φ_4 of formulae (47), we would have, upon eliminating S_4,

$$96\Phi=3S_1^4-15S_1^2S_2+6S_2^2+12S_1S_3. \tag{49}$$

As this expression does not factor, the proof proceeds differently from that for the cases $f=2$, $f=3$. We replace C by CD^{-1} in U and sum for all h elements D. Then

$$\frac{h}{e}[(A)(B)(C)-3(A)(BC)-3(B)(AC)-(C)(AB)+6(ABC)$$
$$+6(ACB)]+h[-2(A)(B)(C)+3(A)(BC)+3(B)(AC)$$
$$+2(C)(AB)-4(ABC)-4(ACB)]+\frac{h}{f}[6(A)(B)(C)-4(A)(BC)$$
$$-4(B)(AC)-8(C)(AB)]=0.$$

* In terms of the roots u_1, u_2, u_3, u_4, of which S_n is the sum of the nth powers, this equation gives $-\Sigma u_i^4+\Sigma u_i^3u_j-\Sigma u_i^2u_ju_i+4u_1u_2u_3u_4=0$, $\Phi=u_1u_2u_3u_4$.

Reducing, and dropping the common factor h, we get

$$(50)\quad \left(\frac{1}{e}-\frac{1}{2}\right)(A)(B)(C)+\left(2-\frac{3}{e}\right)[(A)(BC)+(B)(AC)] - \frac{1}{e}(C)(AB)+\left(\frac{6}{e}-4\right)[(ABC)+(ACB)]=0.$$

Replacing B by BC^{-1} in (49) and summing over all h elements C, we get

$$h\left(\frac{1}{e^2}-\frac{2}{e}+1\right)[(A)(B)-4(AB)]=0.$$

If the last factor vanished identically, we should find, on replacing A by AB^{-1} and summing over all elements B, that $(he^{-1}-4h)(A)=0$, which is impossible. Hence the second factor vanishes, so that $e=1$. Then (50) becomes

$$\tfrac{1}{2}(A)(B)(C)-(A)(BC)-(B)(AC)-(C)(AB)+2(ABC)+2(ACB)=0.$$

Multiplying by $2x_Ax_Bx_C$ and summing over all sets A, B, C, we get

$$S_1^2-6S_1S_2+8S_3=0.$$

Eliminating S_3 from (49), we get $192\Phi=3(S_1^2-2S_2)^2$, contrary to the irreducibility of Φ. Hence $U\neq 0$, so that $e=4=f$.

By a quite similar process, involving however general notations, we obtain, for any f, the alternative $e=f$ or $U=0$, where U is a polynomial in $S_1, S_2, \ldots, S_f$, linear in S_f, and with constant coefficients not all zero. Frobenius shows that such a relation $U=0$ is impossible, since it would lead to an equation of degree f in a certain matrix, whereas the equation of lowest degree satisfied by the latter is of degree $f+1$. Instead of this method of greater theoretical difficulty, we may proceed by the elementary method used above for $f=4$.

Calculation of Group-Characters and Group-Determinants.

14. For $e=f$, equation (23), the first equation (29), and (37), may be written

$$(51)\quad \chi(AB)=\chi(BA),\qquad \sum_R \chi(R)\,\chi(R^{-1})=h,$$
$$h\chi(A)\,\chi(B)=f\sum_R \chi(AR^{-1}BR).$$

Frobenius in his first paper, "Über Gruppencharaktere," and Weber in his *Algebra*, take the relations (51), with the added condition $f > 0$, as the definition of the general character χ of the group H of order h. The third relation, for $B = E$, gives $\chi(e) = f$. This f is called the degree of the character χ. Starting with these definitions, Frobenius shows that f is always a positive integer and that the number of distinct characters is the number r of sets of conjugate elements of the group H. From the standpoint of this paper, these results have been established in §§7, 11. Whether viewed as definitional equations or as derived relations, equations (51) serve to calculate the r distinct characters. We consider some simple examples.

15. Consider the symmetric group on 3 letters and retain the notations of §12. The three sets of conjugate elements are E; A, B; C, D, F. By the first relation (51), $\chi(A) = \chi(B)$, $\chi(C) = \chi(D) = \chi(F)$. By the second and third relations (51), we get

$$6 = f^2 + 3\chi^2(C) + 2\chi^2(A), \quad 6\chi^2(C) = 2f^2 + 4f\chi(A),$$
$$6\chi^2(A) = 3f^2 + 3f\chi(A), \quad 6\chi(A)\,\chi(C) = 6f\chi(C).$$

If $\chi(C) \neq 0$, the last relation gives $\chi(A) = f$, so that the others give

$$\chi^2(C) = f^2, \quad f^2 = 1.$$

If $\chi(C) = 0$, the second relation gives $\chi(A) = -\frac{1}{2}f$, so that the others give $f^2 = 4$. Since $f > 0$, the results for the two cases are as follows:

f	$\chi(E)$	$\chi(A)$	$\chi(C)$
1	1	1	1
1	1	1	−1
2	2	−1	0

(These results agree with the results (42) and (45) of §12). The two characters of degree $f = 1$ lead, in view of formula (8), to the two linear factors (43). To obtain the irreducible factor Φ_2 of degree $f = 2$, we proceed as in §7. Then

$$S_1 = \sum_R \chi(R)x_R = 2x_E - x_A - x_B,$$
$$S_2 = \sum_{R_1, R_2} \chi(R_1R_2)x_{R_1}x_{R_2} = 2(x_E^2 + 2x_Ax_B + x_C^2 + x_D^2 + x_F^2)$$
$$- (2x_Ex_A + x_B^2 + x_Dx_C + x_Fx_D + x_Cx_F)$$
$$- (2x_Ex_B + x_A^2 + x_Fx_C + x_Cx_D + x_Dx_F),$$

the three expressions in parentheses corresponding to the terms of the sum for which $R_1 R_k = E, A, B,$ respectively. Then $2\Phi_2 = S_1^2 - S_2$ leads to the result (44).

16. Consider the dihedron group H_{10} generated by α, β subject to

$$\alpha^5 = E, \quad \beta^2 = E, \quad \alpha\beta = \beta\alpha^{-1} \qquad (E = \text{identity}). \tag{52}$$

The 4 distinct sets of conjugates are E; α, α^4; α^2, α^3; $\beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3, \beta\alpha^4$. The 4 corresponding characters are designated $\chi(E) = f$, $\chi(\alpha) = \chi_1$, $\chi(\alpha^2) = \chi_2$, $\chi(\beta) = \chi_3$. According as $R = \alpha^i$ or $R = \beta\alpha^i$, we have

$$\alpha R^{-1}\alpha R = \begin{cases} \alpha^2 \\ E \end{cases}, \ \alpha^2 R^{-1}\alpha^2 R = \begin{cases} \alpha^4 \\ E \end{cases}, \ \beta R^{-1}\beta R = \begin{cases} \alpha^{2i} \\ \alpha^{2i} \end{cases}, \ \alpha R^{-1}\alpha^2 R = \begin{cases} \alpha^3 \\ \alpha^4 \end{cases},$$

$$\alpha R^{-1}\beta R = \begin{cases} \beta\alpha^{2i-1} \\ \beta\alpha^{2i-1} \end{cases}.$$

Hence the second and third relations (51) give

$$10 = f^2 + 2\chi_1^2 + 2\chi_2^2 + 5\chi_3^2, \quad 10\chi_1^2 = 5f^2 + 5f\chi_2, \quad 10\chi_2^2 = 5f^2 + 5f\chi_1,$$
$$10\chi_3^2 = 2f^2 + 4f\chi_1 + 4f\chi_2, \quad 10\chi_1\chi_2 = 5f\chi_1 + 5f\chi_2, \quad 10\chi_1\chi_3 = 10f\chi_3,$$
$$10\chi_2\chi_3 = 10f\chi_3.$$

If $\chi_3 \neq 0$, the last two relations give $\chi_1 = \chi_2 = f$ and the others then give

$$\chi_3^2 = f^2, \qquad f^2 = 1.$$

Since $f > 0$, we derive $f = 1$, $\chi_1 = \chi_2 = 1$, $\chi_3 = \pm 1$,

If $\chi_3 = 0$, the first four relations give

$$20 = 2f^2 + 4\chi_1^2 + 4\chi_2^2 = 6f^2 + 2f\chi_1 + 2f\chi_2 = 6f^2 - f^2 = 5f^2,$$

whence $f = 2$. The seven relations then reduce to the following:

$$\chi_1^2 = \chi_2 + 2, \ \chi_2^2 = \chi_1 + 2, \ \chi_1 + \chi_2 + 1 = 0, \ \chi_1\chi = \chi_1 + \chi_2,$$

which state merely that χ_1 and χ_2 are the roots of $y^2 + y - 1 = 0$,

$$y_1 = \tfrac{1}{2}(-1 + \sqrt{5}), \quad y_2 = \tfrac{1}{2}(-1 - \sqrt{5}).$$

Hence the four characters are given by the table:

(53)

f	$\chi(E)$	$\chi_1 = \chi(\alpha)$	$\chi_2 = \chi(\alpha^2)$	$\chi_3 = \chi(\beta)$
1	1	1	1	1
1	1	1	1	−1
2	2	y_1	y_2	0
2	2	y_2	y_1	0

Setting $x_0 = x_E$, $x_i = x_{a^i}$ $(i = 1, 2, 3, 4)$, $x'_i = x_{\beta a^i}$ $(i = 0, 1, 2, 3, 4)$, the group-matrix for the dihedron group H_{10} may be written in the abbreviated form

$$\Theta \equiv \left(\begin{array}{c|c} X & X' \\ \hline X' & X \end{array}\right),$$

where

$$X \equiv \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_4 & x_0 & x_1 & x_2 & x_3 \\ x_3 & x_4 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_4 & x_0 \end{pmatrix}, \quad X' \equiv \begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 & x'_4 \\ x'_1 & x'_2 & x'_3 & x'_4 & x'_0 \\ x'_2 & x'_3 & x'_4 & x'_0 & x'_1 \\ x'_3 & x'_4 & x'_0 & x'_1 & x'_2 \\ x'_4 & x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix}$$

In view of (53), there are exactly two linear factors of Θ, viz.,

$$(54) \qquad (x_0 + x_1 + x_2 + x_3 + x_4) \pm (x'_0 + x'_1 + x'_2 + x'_3 + x'_4).$$

The two irreducible factors of the second degree, Φ_2 and Φ'_2, may be calculated by means of the results of §7. Let Φ_2 correspond to the character for which $\chi(a) = y_1$, $\chi(a^2) = y_2$. Then Φ'_2 is derived from Φ_2 by interchanging y_1 and y_2. By (19) and (18),

$$2\,\Phi_2 = S_1^2 - S_2,$$

$$S_1 = \sum_R \chi(R)\, x_R \equiv 2\,x_0 + y_1 x_1 + y_1 x_4 + y_2 x_2 + y_2 x_3,$$

$$\begin{aligned} S_2 = \sum_{R_1, R_2} \chi(R_1 R_2)\, x_{R_1} x_{R_2} \\ \equiv\ & 2(x_0^2 + 2\,x_4 x_1 + 2\,x_3 x_2 + x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2) \\ & + y_1\,(2\,x_0 x_1 + 2\,x_4 x_2 + x_3^2 + x'_0 x'_1 + x'_1 x'_2 + x'_2 x'_3 + x'_3 x'_4 + x'_4 x'_0) \\ & + y_1\,(2\,x_0 x_4 + 2\,x_3 x_1 + x_2^2 + x'_0 x'_4 + x'_1 x'_0 + x'_2 x'_1 + x'_3 x'_2 + x'_4 x'_3) \\ & + y_2\,(2\,x_0 x_2 + 2\,x_4 x_3 + x_1^2 + x'_0 x'_2 + x'_1 x'_3 + x'_2 x'_4 + x'_3 x'_0 + x'_4 x'_1) \\ & + y_2\,(2\,x_0 x_3 + 2\,x_1 x_2 + x_4^2 + x'_0 x'_3 + x'_1 x'_4 + x'_2 x'_0 + x'_3 x'_1 + x'_4 x'_2). \end{aligned}$$

The five expressions in parentheses correspond to the terms of the sum for which $R_1 R_2 = E, a, a^4, a^2, a^3$, respectively. The final result is

$$\begin{aligned} \Phi_2 = {} & x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2 - x_4'^2 + y_1(x_0 x_1 + x_0 x_4 \\ & + x_1 x_2 + x_2 x_3 + x_3 x_4) \\ & - y_1(x'_0 x'_1 + x'_1 x'_2 + x'_2 x'_3 + x'_3 x'_4 + x'_4 x'_0) - y_2(x'_0 x'_2 + x'_1 x'_3 + x'_2 x'_4 + x'_3 x'_0 \\ & + x'_4 x'_1) + y_2(x_0 x_2 + x_0 x_3 + x_1 x_3 + x_1 x_4 + x_2 x_4). \end{aligned}$$

As a first check, we note that the coefficient of $x_E'^{-1}x_R \equiv x_0 x_R$ is $\chi(R)$, as required by §7. As a second check, we verify the results of §9. For the special group-matrix, $x_1 = x_4$, $x_2 = x_3$, $x_0' = x_1' = x_2' = x_3' = x_4'$, so that

$$\Phi_2 = x_0^2 + 2x_1^2 + 2x_2^2 - 5x_0'^2 + y_1(2x_0x_1 + 2x_1x_2 + x_2^2) - 5y_1x_0'^2 - 5y_2x_0'^2$$
$$+ y_2(2x_0x_2 + 2x_1x_2 + x_1^2) \equiv (x_0 + y_1x_1 + y_2x_2)^2 = (\tfrac{1}{2}\Sigma\chi(R)x_R)^2 \equiv v_1^2,$$

upon applying the relations $y_1y_2 = -1$, $y_1 + y_2 = -1$, and formula (35) for $f = 2$. Hence (34) is verified for the case $u = 0$; and, for the general case, by replacing y_E by $y_E + u$.

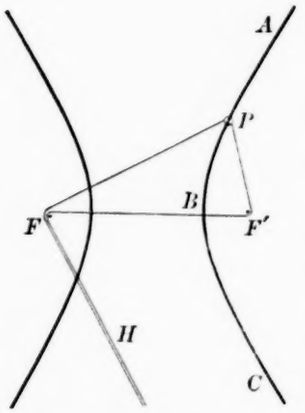
17. A table of characters for each of the groups: Tetrahedron H_{12}, Octahedron H_{24}, Icosahedron H_{60}, symmetric group H_{120} on 5 letters, unary modular group of order $\frac{1}{2}p(p^2-1)$, are given by Frobenius, Über Gruppencharaktere, *l. c.*, pp. 1011–1021. Frobenius has also determined the characters of the symmetric and alternating groups on n letters, giving explicit tables for the case $n = 8$ (*Berliner Sitzungsberichte*, 1900, pp. 516–534; 1901, pp. 303–315). For any symmetric group, all the values of every character are integral rational numbers.

THE UNIVERSITY OF CHICAGO,
MARCH 1902.

COMMUNICATION CONCERNING MR. RANSOM'S MECHANICAL CONSTRUCTION OF CONICS.

To the Editors of the Annals of Mathematics.

Gentlemen:—The following improvement on Mr. Ransom's "mechanical construction of confocal conics,"* was suggested to me by Mr. B. M. Kimball, a member of my Harvard Summer School class in analytic geometry, and may prove of interest to readers of the Annals. Mr. Kimball finds that two pegs, F and F', are sufficient as well for the hyperbola as for the ellipse, the string being placed around the pegs as indicated in the figure.



In the case of the hyperbola, the pencil is fastened to the string at P, so that it will not slip along the string; the loose ends of the string are held by the hand at H, and are drawn downward as P describes the arc AB. When the pencil has reached the position B we may lift it over the string FF' and continue along the arc BC, letting out slack from H; more accurate results will be obtained, however, if the string is first placed around the other side of the peg at F'. The exact position of the hand H is not important, provided both parts of the string always rest firmly against the peg F.

In the case of the ellipse, the point H is held fixed, and the pencil is allowed to slide along the string.

Very truly yours,

Edward V. Huntington.

Harvard University,
August, 1902.

* See the last number of the Annals, p. 164.

CONTENTS.

	PAGE
The Geodesic Lines on the Anchor Ring. By Dr. G. A. Bliss, .	1
Proof of a Theorem Concerning Isosceles Triangles. By Professor H. F. Blichfeldt, ...	22
An Elementary Exposition of Frobenius's Theory of Group-Characters and Group-Determinants. By Professor L. E. Dickson, .	25
Communication concerning Mr. Ransom's Mechanical Construction of Conics. By Dr. E. V. Huntington, ...	50

ANNALS OF MATHEMATICS.

Published in October, January, April, and July, under the auspices of Harvard University, Cambridge, Mass., U. S. A.

Cambridge: Address *The Annals of Mathematics*, 2 University Hall, Cambridge, Mass., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75c. All drafts and money orders should be made payable to Harvard University.

London: Longmans, Green & Co., 39 Paternoster Row. Price, 2 shillings a number.

Leipzig: Otto Harrassowitz, Querstrasse 14. Price, 2 marks a number.

Printed by The Salem Press Co., Salem, Mass., U. S. A.